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Purification of Lindblad dynamics, geometry of mixed states and geometric phases

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ABSTRACT

We propose a nonlinear Schrödinger equation in a Hilbert space enlarged with an ancilla such that the partial trace of its solution obeys to the Lindblad equation of an open quantum system. The dynamics involved by this nonlinear Schrödinger equation constitutes then a purification of the Lindblad dynamics. We study the (non adiabatic) geometric phases involved by this purification and show that our theory unifies several definitions of geometric phases for open systems which have been previously proposed. We study the geometry involved by this purification and show that it is a complicated geometric structure related to a higher gauge theory, i.e. a categorical bibundle with a connective structure.

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1. Introduction

Quantum information [1] and open quantum systems [2] are subjects of particular interest in the modern physics, dealing with decoherence processes, quantum computation and communication, entanglement processes, distillation protocols, Schmidt decomposition, Markovian and non-Markovian effects, etc. A particular interesting subject in quantum information is the process of purification [3] which consists for a mixed state ρ of the Hilbert space $\mathcal{H}_\mathcal{S}$ to find a pure state $\Psi \in \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{A}$ in an enlarged Hilbert space such that $\rho = \mathrm{tr}_{\mathcal{H}_\mathcal{A}} |\Psi\rangle\rangle\langle\langle\Psi|$. The auxiliary Hilbert space $\mathcal{H}_\mathcal{A}$ can be viewed as describing an effective environment. In this paper, we want to study the purification process with respect to the dynamics of mixed states. When the dynamics is conservative, i.e. when it is described by a Liouville–von Neumann equation $\imath\hbar\dot\rho = [H_\mathcal{S}, \rho]$ (no relaxation effect occurs), the dynamics of the purification and the related mathematical structures have been extensively studied, see for example Ref. [4,5]. We study in this paper the case where the environment of the quantum system induces relaxation effects, with a dynamics obeying to a Lindblad equation [2]. We will show that the purified state obeys in the enlarged Hilbert space to a nonlinear Schrödinger equation. The emergence of a nonlinearity is not a new phenomenon in the relation between dynamics of mixed and pure states. In Ref. [6] it is shown that the pure dynamics closest to the Lindblad dynamics (in the sense that this pure dynamics is viewed as the dynamics of some tangent vectors on the density matrix manifold) is nonlinear; and in Ref. [7], a purification protocol needing a nonlinear operation has been proposed (a purification protocol is a set of operations and measurements transforming a mixed state ρ to a pure state ψ of the same Hilbert space without the trace operation – in fact the state ψ depends only on the protocol and is independent from the initial mixed state ρ –, it is a question different from the purification process discussed in the present paper). Moreover the Liouville equation for a piecewise deterministic process is associated for its deterministic part with a nonlinear Schrödinger

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equation but which does not take into account the jump part (see Ref. [2] chapter 6.1) in contrast with our equation for the purified dynamics.

Geometrization of physical theories is a great active area in theoretical physics. In nonrelativistic quantum dynamics, it is in particular related to the theory of geometric phases (so-called Berry phases) [8]. As shown by Simon [9], the dynamics of pure states and the Berry phases take place in a geometric structure which is a principal fibre bundle endowed with a connection. Some generalizations of geometric phases have been proposed for mixed states: Uhlmann [10–12] has proposed a concept of geometric phases based on the theory of transition probabilities for pair of mixed states, the involved geometric structures have been analysed in Ref. [5,13–15]; Sjöqvist et al [16,17] have proposed a concept of geometric phase based on an interferometric theory, the involved geometric structures have been analysed in Ref. [4,18]; and we have proposed a concept of geometric phase in the adiabatic limit based on the generalization of the geometric structure studied by Simon from vector bundles to C^*-modules [19,20]. By using the equation of the purified dynamics, we will build a general theory of geometric phases for open quantum systems which unifies these previous approaches. The dynamics of the density matrices present different geometric phases appearing at different levels. This is due to a more complicated gauge structure called higher gauge theory in the literature [21–25]. We will show that this structure is a generalization in the category theory of the principal bundle structure, complicated by the stratified structure of the density matrix manifold [3].

The approach followed in the present paper could be useful for some problems of quantum control and quantum information concerning open systems. In Ref. [26] we have shown how use the fields associated with the geometry of categorical bundles to analyse the control of a quantum system hampered by entanglement with another one. By the purification of the Lindblad equation, the decoherence and the relaxation effects on the mixed state of the open quantum system, appears in purified picture as entanglement between the quantum state and the ancilla (described by the auxiliary Hilbert space). The fields associated with the categorical bundles presented in this paper, can then be used to analyse the control of an open quantum system (in the purified picture) with a similar method as the one followed in Ref. [26]. Moreover some approaches of quantum computation based on the geometrization of the quantum dynamics and on the geometric phases have been proposed [27–30], usually called holonomic quantum computation (HQC). These approaches are based on the geometric properties of fibre bundles modelizing the dynamics of closed quantum systems. The present work, with the construction of the categorical bundles describing open quantum systems in the purified picture, is a first step to a generalization of the HQC taking into account the decoherence and the relaxation effects occurring for the open systems.

This paper is organized as follows. Section 2 is devoted to the geometry associated with the purification process by recalling the stratified structure of the density matrix manifold and by introducing some representations of the purified states with the associated inner products. This section introduces some mathematical tools needed to the understanding of the following. Section 3 shows that the purified state satisfies a nonlinear Schrödinger equation if the associated mixed state satisfies a Lindblad equation. Section 4 presents a general theory of geometric phases for open quantum systems; there relations with the different propositions of geometric phases are analysed. Section 5 studies the geometric structure involved by the general geometric phase theory and the purification process, firstly from the point of view of the ordinary differential geometry, secondly from the point of view of the category theory. It concludes by the introduction of the connective structure and the physical interpretations of the different fields involved by the connection.

A note about the notations used here:

We adopt the Einstein's convention: a bottom–top repetition of an index induces a summation.

$\mathcal{B}(\mathcal{H})$ *denotes the set of the bounded linear operators of the Hilbert space* $\mathcal{H}$. *For an operator* $A \in \mathcal{B}(\mathcal{H})$, $\mathrm{Ran}A$, $\ker A$ *and* $\mathrm{Sp}(A)$ *denote its range, kernel and spectrum.* $\forall A, B \in \mathcal{B}(\mathcal{H})$, *we denote the commutator and the anticommutator by* $[A, B] = AB - BA$ *and* $\{A, B\} = AB + BA$. $\mathrm{Aut}\mathfrak{g}$, *with* $\mathfrak{g}$ *a vector space or an algebra, denotes the set of the automorphisms of* $\mathfrak{g}$.

The symbol "$\simeq$*" between two spaces (resp. manifolds) denotes that they are isomorphic (resp. homeomorphic). The symbol "*$\approx$*" between two manifolds denotes that they are locally homeomorphic. The symbol "*$A \hookrightarrow B$*" denotes an inclusion of A into B.* $G \rtimes H$ *denotes a semi-direct product between two groups G and H;* $\mathfrak{g} \ni \mathfrak{h}$ *denotes a semi-direct sum between two algebras* $\mathfrak{g}$ *and* $\mathfrak{h}$.

$\mathrm{Pr}_i : V_1 \times V_2 \times \cdots \times V_n \to V_i$, *with* V_j *some sets, denotes the canonical projection* $\mathrm{Pr}_i(x_1, x_2, \ldots, x_n) = x_i$. *Let M be a manifold,* T_xM *denotes its tangent space at x (TM denotes its tangent bundle) and* $\Omega^n(M, \mathfrak{g})$ *denotes its space of* $\mathfrak{g}$*-valued differential n-forms. Let* $f : M \to N$ *be a diffeomorphism between two manifolds,* $f_* : TM \to TN$ *denotes its tangent map (its push-forward) and* $f^* : \Omega^*N \to \Omega^*M$ *denotes its cotangent map (its pull-back).* Let $E \xrightarrow{\pi} M$ be a fibre bundle (E and M are manifolds and π is a surjective map); $\Gamma(M, E)$ denotes the set of its local sections.

For a category $\mathscr{C}$, $\mathrm{Obj}\mathscr{C}$ *denotes its collection of objects and* $\mathrm{Morph}\mathscr{C}$ *denotes its collection of arrows (morphisms).* $\forall o \in \mathrm{Obj}\mathscr{C}$, id_o *denotes the trivial arrow from and to o (the identity map of o).* $\forall a \in \mathrm{Morph}\mathscr{C}$, $s(a)$ *denotes the source of a, and* $t(a)$ *denotes the target of a.* $\mathrm{Funct}(\mathscr{C}, \mathscr{C}')$ *denotes the set of functors from* $\mathscr{C}$ *to* $\mathscr{C}'$ $(\mathrm{EndFunct}(\mathscr{C}) \equiv \mathrm{Funct}(\mathscr{C}, \mathscr{C}))$.

$\forall A(t) \in \mathcal{B}(\mathcal{H})$, $\underset{\leftarrow}{\mathbb{T}e}^{-\int_{t_0}^{t} A(t')dt'} = U_A(t, t_0)$ *denotes the time-ordered exponential (the Dyson series) i.e. the solution of the equation:* $\frac{dU_A(t,t_0)}{dt} = -A(t)U_A(t, t_0)$ *(with* $U_A(t_0, t_0) = \mathrm{id}_{\mathcal{H}}$*).* $\underset{\rightarrow}{\mathbb{T}e}^{-\int_{t_0}^{t} A(t')dt'} = V_A(t_0, t)$ *denotes the time-anti-ordered exponential, i.e. the solution of the equation:* $\frac{dV_A(t_0,t)}{dt} = -V_A(t_0, t)A(t)$ *(with* $V_A(t_0, t_0) = \mathrm{id}_{\mathcal{H}}$*).*

2. Purification process

2.1. The purification bundle and the stratification

Let $\mathcal{H}_\mathcal{S}$ be the Hilbert space of the studied system $\mathcal{S}$ (we consider that $\mathcal{H}_\mathcal{S}$ is finite dimensional, $\mathcal{H}_\mathcal{S} \simeq \mathbb{C}^n$, since the applications of the present work concern essentially quantum information theory where the qubit state spaces are finite

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