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Compatibility of Riemannian structures and Jacobi structures ☆

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We give a notion of compatibility between a Riemannian structure and a Jacobi structure. We prove that in case of fundamental examples of Jacobi structures : Poisson structures, contact structures and locally conformally symplectic structures, we get respectively Riemann–Poisson structures in the sense of M. Boucetta, (1/2)-Kenmotsu structures and locally conformally Kähler structures.

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1. Introduction

The notion of Jacobi manifold was introduced separately by A. Kirillov, as a local Lie algebra structure on the sections of the trivial line bundle on the underlying manifold [1], and by A. Lichnerowicz who viewed it as a contravariant generalization of the notion of contact manifold [2]. Jacobi manifolds generalize at the same time Poisson manifolds, which are very useful in classical analytical mechanics and in quantum mechanics, and contact manifolds which are especially used in the geometrical study of partial differential equations. In [3], P. Dazord, A. Lichnerowicz and C.-M. Marle studied the local structure of a Jacobi manifold, it follows in particular that a regular transitive Jacobi manifold is either contact or locally conformally symplectic according to the dimension of the underlying manifold.

The presence of a pseudo-Riemannian metric on a Jacobi manifold induces in certain special cases and under certain conditions remarkable geometric structures. On one hand, J. Rakotondralambo [4], defined a notion of compatibility between a pseudo-Riemannian metric and a symplectic form which in the case of an almost Hermitian manifold induces a Kähler structure. Then, in [5,6], M. Boucetta generalized this notion to a Poisson manifold to get a structure that he called a pseudo-Riemannian Poisson structure. He proved that the symplectic leaves of a regular Riemannian Poisson manifold are Kähler. On the other hand, the data of a pseudo-Riemannian metric and an almost contact structure on a manifold give rise to interesting geometrical structures like Sasakian manifolds and Kenmotsu manifolds, see [7].

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One of the natural questions posing itself is the one of a good notion of compatibility between a pseudo-Riemannian metric and a Jacobi structure. In this work, we introduce such a notion which in the case of a Poisson manifold gives a pseudo-Riemannian Poisson structure. We also prove that for a contact Riemannian structure, with this notion of compatibility we get a (1/2)-Kenmotsu structure, and that in the case of a locally conformally symplectic structure with an "associated" metric, we get a locally conformal Kähler structure.

Let us summarize the contents of this paper. Let M be a smooth manifold. We consider on M a bivector field π, a vector field ξ and a 1-form λ, and associate with the triple (π, ξ, λ) a skew algebroid $(T^*M, \sharp_{\pi,\xi}, [.,.]^\lambda_{\pi,\xi})$ on M. We prove that if the pair (π, ξ) is a Jacobi structure and that $\pi \neq 0$, the skew algebroid $(T^*M, \sharp_{\pi,\xi}, [.,.]^\lambda_{\pi,\xi})$ is an almost Lie algebroid if and only if $\sharp_{\pi,\xi}(\lambda) = \xi$. In case $\xi = \lambda = 0$, it is the cotangent algebroid of the Poisson manifold (M, π). We also prove that in case (π, ξ) is the Jacobi structure associated with a contact form η, respectively with a locally conformally symplectic structure (ω, θ), the skew algebroid $(T^*M, \sharp_{\pi,\xi}, [.,.]^\eta_{\pi,\xi})$, respectively $(T^*M, \sharp_{\pi,\xi}, [.,.]^\theta_{\pi,\xi})$, is a Lie algebroid isomorphic to the tangent algebroid of M.

Next, for a triple (π, ξ, g) consisting of a bivector field π, a vector field ξ and a pseudo-Riemannian metric g on M, we put $\lambda = g(\xi,\xi)\flat_g(\xi) - \flat_g(J\xi)$ and $[.,.]^g_{\pi,\xi} = [.,.]^\lambda_{\pi,\xi}$, where $\flat_g : TM \to T^*M$ and $\sharp_g = \flat_g^{-1}$ are the musical isomorphisms of g and where J is the endomorphism of the tangent bundle TM given by $\pi(\alpha, \beta) = g(J\sharp_g(\alpha), \sharp_g(\beta))$, and define a contravariant derivative $\mathcal{D}$ to be the unique contravariant symmetric derivative compatible with g. If (π, ξ) is Jacobi, and if $\sharp_{\pi,\xi}$ is an isometry, a condition that is satisfied in the particular cases of a contact and of a locally conformally symplectic structure, we prove that $\mathcal{D}$ is related to the (covariant) Levi-Civita connection ∇ of g by $\sharp_{\pi,\xi}(\mathcal{D}_\alpha\beta) = \nabla_{\sharp_{\pi,\xi}(\alpha)}\sharp_{\pi,\xi}(\beta)$.

Finally, with the use of the contravariant Levi-Civita derivative $\mathcal{D}$ we introduce a notion of compatibility of the triple (π, ξ, g). In case $\xi = 0$, it is just the compatibility of the pair (π, g) introduced by M. Boucetta [5]. In the case of a Jacobi structure (π, ξ) associated with a contact metric structure (η, g), the triple (π, ξ, g) is compatible if and only if the structure (η, g) is (1/2)-Kenmotsu. In case (π, ξ) is the Jacobi structure associated with a locally conformally symplectic structure (ω, θ), if g is a somehow associated metric, the triple (π, ξ, g) is compatible if and only if the structure (ω, θ, g) is locally conformally Kähler.

2. Almost Lie algebroids associated with a Jacobi manifold

2.1. Almost Lie algebroids associated with a Jacobi manifold

Throughout this paper M is a smooth manifold, π a bivector field and ξ a vector field on M.

The pair (π, ξ) defines a Jacobi structure on M if we have the relations

$$[\pi, \pi] = 2\xi \wedge \pi \quad \text{et} \quad [\xi, \pi] := \mathcal{L}_\xi \pi = 0, \tag{2.1}$$

where $[.,.]$ is the Schouten–Nijenhuis bracket. We say that (M, π, ξ) is a Jacobi manifold. In the case $\xi = 0$, the relations above are reduced to $[\pi, \pi] = 0$ that corresponds to a Poisson structure (M, π).

Recall on the other hand, see for instance [8], that a skew algebroid over M is a triple $(A, \sharp_A, [.,.]_A)$ where A is the total space of a vector bundle on M, $\sharp_A$ is a vector bundle morphism from A to TM, called the anchor map, and $[.,.]_A : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A), (a, b) \longmapsto [a, b]_A$, is an alternating $\mathbb{R}$-bilinear map over the space $\Gamma(A)$ of sections of A verifying the Leibniz identity:

$$[a, \varphi b]_A = \varphi[a, b]_A + \sharp_A(a)(\varphi)b, \quad \forall \varphi \in C^\infty(M),\ \forall a, b \in \Gamma(A).$$

A skew algebroid $(A, \sharp_A, [.,.]_A)$ is an almost Lie algebroid if

$$\sharp_A([a, b]_A) = [\sharp_A(a), \sharp_A(b)], \quad \forall a, b \in \Gamma(A),$$

and a Lie algebroid if $(\Gamma(A), [.,.]_A)$ is a Lie algebra, i.e., if the bracket $[.,.]_A$ satisfy the Jacobi identity

$$[a, [b, c]_A]_A + [b, [c, a]_A]_A + [c, [a, b]_A]_A = 0, \quad \forall a, b, c \in \Gamma(A).$$

Note that a Lie algebroid is an almost Lie algebroid, indeed, it is well known that the Leibniz identity and the Jacobi identity together imply that the anchor map is a Lie algebra morphism. On the other hand, an almost Lie algebroid $(A, \sharp_A, [.,.]_A)$ such that the anchor map $\sharp_A$ is an isomorphism is a Lie algebroid isomorphic to the tangent algebroid $(TM, \mathrm{id}_M, [.,.])$ of M.

Let $\sharp_\pi : T^*M \to TM$ be the vector bundle morphism defined by $\beta(\sharp_\pi(\alpha)) = \pi(\alpha, \beta)$ and let $[.,.]_\pi : \Omega^1(M) \times \Omega^1(M) \longrightarrow \Omega^1(M)$ be the map defined by

$$[\alpha, \beta]_\pi := \mathcal{L}_{\sharp_\pi(\alpha)}\beta - \mathcal{L}_{\sharp_\pi(\beta)}\alpha - d(\pi(\alpha, \beta)),$$

called the Koszul bracket. Consider the morphism of vector bundles $\sharp_{\pi,\xi} : T^*M \longrightarrow TM$ defined by

$$\sharp_{\pi,\xi}(\alpha) = \sharp_\pi(\alpha) + \alpha(\xi)\xi$$

and, for a 1-form $\lambda \in \Omega^1(M)$, the map $[.,.]^\lambda_{\pi,\xi} : \Omega^1(M) \times \Omega^1(M) \longrightarrow \Omega^1(M)$ defined by

$$[\alpha, \beta]^\lambda_{\pi,\xi} := [\alpha, \beta]_\pi + \alpha(\xi)\left(\mathcal{L}_\xi\beta - \beta\right) - \beta(\xi)\left(\mathcal{L}_\xi\alpha - \alpha\right) - \pi(\alpha, \beta)\lambda.$$

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