Journal of Geometry and Physics 132 (2018) 13–44

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys

Killing superalgebras for lorentzian six-manifolds

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ARTICLE INFO

Article history:
Received 4 April 2018
Received in revised form 5 May 2018
Accepted 19 May 2018
Available online 5 June 2018

Keywords:
Killing superalgebras
Rigid supersymmetry
Spencer cohomology

ABSTRACT

We calculate the Spencer cohomology of the $(1, 0)$ Poincaré superalgebras in six dimensions: with and without R-symmetry. As the cases of four and eleven dimensions taught us, we may read off from this calculation a Killing spinor equation which allows the determination of which geometries admit rigidly supersymmetric theories in this dimension. We prove that the resulting Killing spinors generate a Lie superalgebra and determine the geometries admitting the maximal number of such Killing spinors. They are divided in two branches. One branch consists of the lorentzian Lie groups with bi-invariant metrics and, as a special case, it includes the lorentzian Lie groups with a self-dual Cartan three-form which define the maximally supersymmetric backgrounds of $(1, 0)$ Poincaré supergravity in six dimensions. The notion of Killing spinor on the other branch does not depend on the choice of a three-form but rather on a one-form valued in the R-symmetry algebra. In this case, we obtain three different (up to local isometry) maximally supersymmetric backgrounds, which are distinguished by the causal type of the one-form.

1. Introduction

There has been considerable interest over recent years in the systematic exploration of curved backgrounds that support some amount of rigid (conformal) supersymmetry. The primary motivation being that quantum field theories on such backgrounds are often amenable to the powerful techniques of supersymmetric localisation, typically revealing interesting new insights and exact results [1–9].

By far the most successful strategy in this direction was initiated by Festuccia and Seiberg [10], originally for rigidly supersymmetric backgrounds in four dimensions but subsequently generalised [11–18] to other dimensions in both euclidean and lorentzian signatures. Their method takes advantage of the existence of some locally supersymmetric supergravity theory coupled to one or more field theory supermultiplets. In any such theory, it is possible to take a certain rigid limit in which the Planck mass tends to infinity and the degrees of freedom from the gravity supermultiplet are effectively frozen out. What remains after taking this limit is a rigidly supersymmetric field theory on a bosonic supersymmetric background of the original supergravity theory. The Killing spinor equations which characterise this supersymmetric background are simply read off from the supersymmetry variation of the gravitino in the rigid limit. It is important to emphasise that these supersymmetric backgrounds need not solve the supergravity field equations. For example, in four dimensions, the old minimal off-shell formulation of Poincaré supergravity contains auxiliary fields which are all set to zero by the field equations. However, many interesting rigidly supersymmetric backgrounds of this theory are not solutions because they are supported by one or more non-zero auxiliary fields [10,19].

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https://doi.org/10.1016/j.geomphys.2018.05.019
0393-0440/

The precise details of the rigid supersymmetry supported by any bosonic supersymmetric supergravity background are encoded by its Killing superalgebra [20–25]. The Killing superalgebra is a Lie superalgebra whose odd part consists of all the Killing spinors supported by the background and whose even part contains Killing vectors which preserve the background. For supergravity theories with a non-trivial R-symmetry, the even part of the Killing superalgebra may also contain R-symmetries which preserve the background. While the appearance of the Killing superalgebra may seem somewhat peripheral in relation to the rigid limit described above, it is clearly an object of fundamental significance in the description of rigid supersymmetry and in understanding special geometrical properties of the backgrounds which support it.

So much so that, somewhat in the spirit of the Erlangen program, one might prefer to take the classification of Killing superalgebras as the central question, with no prior knowledge of supergravity, and then deduce as a by product all the possible rigidly supersymmetric backgrounds (which may or may not correspond to backgrounds of some known supergravity theory). This is certainly the philosophy we have adopted in some of our previous works, which has led to the classification of Killing superalgebras for maximally supersymmetric lorentzian backgrounds in dimensions eleven [26,27] and four [19]. The key property of Killing superalgebras that permits such a classification is the fact that they are all filtered deformations (in a certain technical sense which we review in Section 8.5) of some subalgebra of the Poincaré superalgebra, possibly extended by R-symmetries. As one might expect, there is a natural cohomology theory (a generalised version of Spencer cohomology) which governs these filtered deformations at the infinitesimal level, and the essence of the classification is the calculation of a certain Spencer cohomology group in degree two. In dimensions eleven and four [19,26,27], this calculation actually prescribes a Killing spinor equation which is in precise agreement with the Killing spinor equation that characterises bosonic supersymmetric backgrounds of minimal Poincaré supergravity in these respective dimensions (more accurately, in the 'old minimal' off-shell formulation in four dimensions [19]). So, at least in these cases, all the rigidly supersymmetric backgrounds are indeed backgrounds of a known Poincaré supergravity theory.

In this paper, we shall extend these considerations to look at Killing superalgebras for lorentzian backgrounds in six dimensions. There are several reasons that make dimension six especially interesting. Recall that the Lie superalgebra $\mathfrak{osp}(6,2|N)$ is isomorphic to the N-extended conformal superalgebra of $\mathbb{R}^{5,1}$, and that conformal superalgebras do not exist in higher dimensions (at least, not in the traditional sense of Nahm [28]). Furthermore the $\mathfrak{sp}(N)$ R-symmetry subalgebra of $\mathfrak{osp}(6,2|N)$ is nonabelian, for any $N > 0$. Now let $N = 1$. By omitting the dilatations and special conformal transformations in the even part of $\mathfrak{osp}(6,2|1)$, together with the special conformal supercharges in the odd part, we obtain a Lie superalgebra that we will denote by $\hat{\mathfrak{p}}$. If we also omit the $\mathfrak{sp}(1)$ R-symmetry in $\hat{\mathfrak{p}}$, we recover the ordinary $(1,0)$ Poincaré superalgebra in six dimensions (without R-symmetry) that we will denote by $\mathfrak{p}$.

Our first goal in this paper will be to calculate the relevant Spencer cohomology groups for both $\mathfrak{p}$ and $\hat{\mathfrak{p}}$, see Theorems 10 and 11. In marked contrast with the situation in dimensions eleven and four, where the inclusion of R-symmetry is immaterial, here in dimension six we will find that the relevant Spencer cohomology groups for $\mathfrak{p}$ and $\hat{\mathfrak{p}}$ are different. In both cases, we then go on to use the explicit expression for a particular component of the Spencer cocycle representative to prescribe an appropriate Killing spinor equation. For the case where the R-symmetry is not gauged, this Killing spinor equation is given by Definition 12 in Section 6 while, for the case of gauged R-symmetry, the Killing spinor equation is defined by (157) in Section 7. In both cases, on a six-dimensional spin manifold M equipped with a lorentzian metric g, we find that the extra background data needed to define this Killing spinor equation consists of a three-form H and an $\mathfrak{sp}(1)$-valued one-form φ. (For the case of gauged R-symmetry, one must also specify a flat $\mathfrak{sp}(1)$ connection C.) The important distinction is that H must be self-dual for $\mathfrak{p}$ whereas, for $\hat{\mathfrak{p}}$, its anti-self-dual component H^- need not be zero. It is important to stress that the Killing spinor equation we deduce from Spencer cohomology agrees with the Killing spinor equation for bosonic supersymmetric backgrounds of $(1,0)$ Poincaré supergravity in six dimensions *only* when H is self-dual and $\varphi = 0$. (For the case of gauged R-symmetry, one can remove the flat connection C by an appropriate choice of gauge.) It is an intriguing question as to whether our more general Killing spinor equation can be recovered from supergravity, perhaps via superconformal compensators.

We then proceed to the construction of Killing superalgebras based on these Killing spinors. The geometric content of the cocycle conditions for Spencer cohomology is that the Dirac current of a Killing spinor (derived from Spencer cohomology) is a Killing vector and that the Lie derivative along the Dirac current annihilates the Killing spinor itself. In order to prove that Killing spinors generate a Lie superalgebra, the only additional requirement is that the Lie derivative along the Dirac current of any Killing spinor preserves the space of Killing spinors. This is guaranteed if the connection $\mathscr{D}$ defining the notion of a Killing spinor is invariant along the flow generated by the Dirac current of any Killing spinor.

An equivalent condition for the invariance of $\mathscr{D}$ is the invariance of the other geometric data defining $\mathscr{D}$: the three-form H and $\mathfrak{sp}(1)$-valued one-form φ. For $\mathfrak{p}$, we establish the existence of a Killing superalgebra provided H is closed and φ is coclosed, see Theorem 20. For $\hat{\mathfrak{p}}$, if $\varphi = 0$, we find that a Killing superalgebra exists provided H is closed and H^- is parallel with respect to the metric connection with skew-symmetric torsion given by H^+, see Theorem 24.

Finally, we present in Theorem 27 the classification (up to local isometry) of all backgrounds which admit the maximal number of Killing spinors. In addition to Minkowski space $\mathbb{R}^{5,1}$, we find that there are two distinct branches of maximally supersymmetric backgrounds. All backgrounds on the first branch are conformally flat and have $H = 0$ with $\varphi = a \otimes R$, where a is a non-zero parallel one-form and R is a non-zero element of $\mathfrak{sp}(1)$. Up to local isometry, there are three different backgrounds on this branch which depend only on the causal type of a:

- $\mathrm{AdS}_5 \times \mathbb{R}$, if a is spacelike;