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On the geometrical interpretation of locality in anomaly cancellation

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ABSTRACT

A notion of local section of the determinant line bundle is defined giving necessary and sufficient conditions for anomaly cancellation compatible with locality. This definition gives an intrinsic geometrical interpretation of the local counterterms allowed in the renormalization program of quantum field theory. For global anomalies the conditions for anomaly cancellation are expressed in terms of the equivariant holonomy of the Bismut–Freed connection.

1. Introduction

Anomalies in quantum field theory admit a geometrical interpretation in terms of determinant (or Pfaffian) line bundles (e.g. see [1–3]). In order to have a well defined quantum field theory the determinant line bundle should be trivial. However, it is well known that this condition is not sufficient for anomaly cancellation due to the locality problem. Hence, to cancel the anomaly the determinant line bundle should admit a special kind of section (a local section) corresponding to the local counterterms allowed in the renormalization program of quantum field theory. In this paper we study the geometrical interpretation of these local sections in terms of the Bismut–Freed connection and we obtain necessary and sufficient conditions for perturbative and global anomaly cancellation.

Let us explain in more detail the locality problem. We consider the action of a group $\mathcal{G}$ on a bundle $E \to M$ over a compact n-manifold M. Let $\{D_s : s \in \Gamma(E)\}$ be a $\mathcal{G}$-equivariant family of elliptic operators acting on chiral fermionic fields $\psi \in \Gamma(V)$ and parametrized by $\Gamma(E)$. For example, for gravitational anomalies $\Gamma(E)$ is the space of Riemannian metrics, $\mathcal{G}$ is the diffeomorphisms group of M and D_s is the Dirac operator. For gauge anomalies $\Gamma(E)$ is the space of connections on a principal bundle $P \to M$, $\mathcal{G}$ is the group of gauge transformations or the automorphisms group of P, E is the bundle of connections on P and D_s the Dirac operator coupled to a connection on P (see e.g. [4] for details).

Then the Lagrangian density $\lambda_D(\psi, s) = \bar{\psi} i D_s \psi$ is $\mathcal{G}$-invariant, and hence the classical action $\mathcal{A}_{\mathcal{L}}(\psi, s) = \int_M \bar{\psi} i D_s \psi$, is a $\mathcal{G}$-invariant function on $\Gamma(V) \times \Gamma(E)$. However, at the quantum level, the corresponding partition function defined by a formal fermionic path integral by $\mathcal{Z}(s) = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left(-\int_M \bar{\psi} i D_s \psi\right)$ could fail to be $\mathcal{G}$-invariant. $\mathcal{Z}(s)$ can be defined in terms of regularized determinants of elliptic operators but not in a unique way. There is an ambiguity in the definition of

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$\mathcal{Z}(s)$ modulo the addition of local counterterms (e.g. see [5]). Due to this ambiguity $\mathcal{Z}(s)$ could fail to be $\mathcal{G}$-invariant. It can be seen that the modulus of $\mathcal{Z}(s)$ is $\mathcal{G}$-invariant. Hence we have $\mathcal{Z}(\phi\cdot s)=\mathcal{Z}(s)\cdot\exp(2\pi i\cdot\alpha_{\phi}(s))$ where $\alpha:\mathcal{G}\times\Gamma(E)\to\mathbb{R}/\mathbb{Z}$ satisfies the cocycle condition $\alpha_{\phi_2\phi_1}(s)=\alpha_{\phi_1}(s)+\alpha_{\phi_2}(\phi_1 s)$. Different definitions of $\mathcal{Z}(s)$ determine different cocycles, but they are cohomologous, in the sense that they satisfy the condition $\alpha'_{\phi}=\alpha_{\phi}+\phi^*\theta-\theta$ for some $\theta\in\Omega^0(\Gamma(E))$. If α is an exact cocycle, i.e., if there exists $\Lambda\in\Omega^0(\Gamma(E))$ satisfying

$$\alpha_{\phi}(s)=\Lambda(\phi\cdot s)-\Lambda(s) \tag{*}$$

we can define $\mathcal{Z}'=\mathcal{Z}\cdot\exp(-2\pi i\Lambda)$ and we have $\mathcal{Z}'(\phi\cdot s)=\mathcal{Z}'(s)$. Hence the anomaly can be represented by a cohomology class in $H^1(\mathcal{G},\Omega^0(\Gamma(E),\mathbb{R}/\mathbb{Z}))\simeq H^1(\mathcal{G},\Omega^0(\Gamma(E))/\mathbb{Z})$ (e.g. see [6–8]). For perturbative anomalies the group cohomology can be replaced by Lie algebra cohomology. For $X\in\mathrm{Lie}\mathcal{G}$ we define $\mathfrak{a}(X)=\left.\frac{\delta\alpha_{\phi_t}}{\delta t}\right|_{t=0}$ with $\phi_t=\exp(tX)$. If $\mathcal{G}$ is connected condition (*) is equivalent to $\mathfrak{a}(X)=L_X\Lambda$. Hence the condition for perturbative anomaly cancellation is equivalent to $[\mathfrak{a}]=0$ on $H^1(\mathrm{Lie}\mathcal{G},\Omega^0(\Gamma(E)))$ ($\mathfrak{a}$ is closed by the Wess–Zumino consistency condition).

However, from the physical point of view that is not the end of the story. Physics require that $\mathcal{Z}'$ should be the fermionic path integral of a Lagrangian density, and hence $\Lambda(s)$ should be a *local* functional, i.e., it should be of the form $\Lambda(s)=\int_M\lambda(s)$, where $\lambda(s)(x)$ is a function of $s(x)$ and the derivatives of s at x. If that is the case, we can modify the Lagrangian density to the effective Lagrangian $\mathcal{L}'(s)=\bar{\psi}iD_s\psi-\lambda(s)$ and the partition function of $\mathcal{L}'$ is $\mathcal{Z}'$. We say that the topological anomaly cancels if condition (*) is satisfied for a functional $\Lambda\in\Omega^0(\Gamma(E))$, and that the physical anomaly cancels if condition (*) is satisfied for a local functional $\Lambda\in\Omega^0_{\mathrm{loc}}(\Gamma(E))$. Obviously the second condition implies the first, but the converse is not true. Furthermore, if condition (*) is satisfied only for the connected component of the identity $\mathcal{G}_0$ on $\mathcal{G}$ we say that the perturbative (or local) anomaly cancels. If it is satisfied for all the elements of $\mathcal{G}$ we say that the global anomaly cancels. Hence the perturbative physical anomaly is represented by a cohomology class in the local BRST cohomology $H^1(\mathrm{Lie}\mathcal{G},\Omega^0_{\mathrm{loc}}(\Gamma(E)))$ defined in [9], and the global physical anomaly by a class in $H^1(\mathcal{G},\Omega^0_{\mathrm{loc}}(\Gamma(E))/\mathbb{Z})$.

The condition (*) admits the following geometrical interpretation. The cocycle α determines an action on the trivial bundle $\mathcal{L}=\Gamma(E)\times\mathbb{C}\to\Gamma(E)$ by setting $\phi_{\mathcal{U}}(s,u)=(\phi(s),u\cdot\exp(2\pi i\alpha_{\phi}(s)))$ for $s\in\Gamma(E)$ and $u\in\mathbb{C}$. If the action of $\mathcal{G}$ on $\Gamma(E)$ is free we can consider the quotient bundle $\underline{\mathcal{L}}=(\Gamma(E)\times\mathbb{C})/\mathcal{G}\to\Gamma(E)/\mathcal{G}$, and $\mathcal{Z}$ determines a section of $\underline{\mathcal{L}}$. Furthermore, if Λ satisfies Eq. (*), then $\exp(2\pi i\Lambda)$ determines a section of unitary norm of $\underline{\mathcal{L}}$, i.e., a section of the principal $U(1)$-bundle $\underline{\mathcal{U}}=(\Gamma(E)\times U(1))/\mathcal{G}\to\Gamma(E)/\mathcal{G}$. Hence topological anomaly cancellation is equivalent to the existence of a section of $\underline{\mathcal{U}}\to\Gamma(E)/\mathcal{G}$, and hence to the triviality of $\underline{\mathcal{L}}$.

In [2] the bundle $\underline{\mathcal{L}}$ is identified with the determinant line bundle of the family of operators. If $\mathcal{G}$ is connected (i.e. for perturbative anomalies) a necessary and sufficient condition for topological anomaly cancellation is that $c_1(\underline{\mathcal{L}})=0$ on $H^2(\Gamma(E)/\mathcal{G})$. The advantage of this approach to anomalies is that the Atiyah–Singer Index Theorem gives an explicit expression for $c_1(\underline{\mathcal{L}})$ in terms of characteristic classes. Furthermore, it gives the curvature $\mathrm{curv}(\underline{\Xi})$ of the Bismut–Freed connection $\underline{\Xi}$ on $\underline{\mathcal{L}}$ (e.g. see [10]). Note the difference with the approach based on group and Lie algebra cohomology, where α and $\mathfrak{a}$ are defined only modulo exact terms and given by complicated expressions on secondary characteristic classes. This approach also gives a geometrical interpretation of the anomaly as a cohomology class in $\Gamma(E)/\mathcal{G}$, and allows the use of topological tools in the study of anomaly cancellation. However, due to the locality problem, the cancellation of topological anomalies only gives necessary conditions for physical anomaly cancellation, but they are not sufficient, i.e., anomalies in field theory can exist even if the corresponding topological anomaly is trivial. In order to take into account locality, it is proposed in [11] (see also [1]) the problem of defining a notion of "local cohomology" giving necessary and sufficient condition for physical anomaly cancellation. This problem was solved in [4] for perturbative anomalies by the introduction of local equivariant cohomology. In place of working with the cohomology of the quotient $H^2(\Gamma(E)/\mathcal{G})$ we can also consider the $\mathcal{G}$-equivariant cohomology $H^2_{\mathcal{G}}(\Gamma(E))$. For free actions we have $H^2(\Gamma(E)/\mathcal{G})\simeq H^2_{\mathcal{G}}(\Gamma(E))$, and we can consider $\mathcal{L}=\Gamma(E)\times\mathbb{C}\to\Gamma(E)$ as a $\mathcal{G}$-equivariant line bundle and $\mathcal{U}=\Gamma(E)\times U(1)\to\Gamma(E)$ as a $\mathcal{G}$-equivariant $U(1)$-bundle. Furthermore, the $\mathcal{G}$-equivariant curvature $\mathrm{curv}_{\mathcal{G}}(\Xi)$ of the Bismut–Freed connection Ξ on $\mathcal{L}$ is given by the equivariant Atiyah–Singer Index Theorem (see [12]). One of the advantages of equivariant cohomology is that it is also well defined for non-free actions. But the most important advantage of $H^2_{\mathcal{G}}(\Gamma(E))$ with respect to $H^2(\Gamma(E)/\mathcal{G})$ is that $\mathrm{curv}_{\mathcal{G}}(\Xi)$ is a *local form*, whereas $\mathrm{curv}(\underline{\Xi})$ is non-local. In [4] the notions of local forms $\Omega^{\bullet}_{\mathrm{loc}}(\Gamma(E))$ and local equivariant forms $\Omega^{\bullet}_{\mathrm{loc},\mathcal{G}}(\Gamma(E))$ are defined in terms of the jet bundle of E. For Gauge and gravitational anomalies we have $\mathrm{curv}_{\mathcal{G}}(\Xi)\in\Omega^2_{\mathrm{loc},\mathcal{G}}(\Gamma(E))$. Furthermore, the cancellation of the class of $\mathrm{curv}_{\mathcal{G}}(\Xi)$ on $H^2_{\mathrm{loc},\mathcal{G}}(\Gamma(E))$ is equivalent to the cancellation of the perturbative physical anomaly. This approach provides new techniques for the study of anomaly cancellation as the local cohomology $H^2_{\mathcal{G},\mathrm{loc}}(\Gamma(E))$ is very different to the cohomology $H^2(\Gamma(E)/\mathcal{G})$ of the quotient space. It is shown in [13] and [4] that $H^2_{\mathcal{G},\mathrm{loc}}(\Gamma(E))$ is related to the equivariant cohomology of jet bundles and Gelfand–Fuks cohomology of formal vector fields.

The objective of this paper is to give a geometrical interpretation of the preceding results and to generalize the results of [4] to global anomalies. Our starting point for the study of anomaly cancellation is the unitary determinant bundle $\mathcal{U}\to\Gamma(E)$ corresponding to a $\mathcal{G}$-equivariant family of elliptic operators [10]. We consider $\mathcal{U}\to\Gamma(E)$ as a $\mathcal{G}$-equivariant $U(1)$-bundle and the Bismut–Freed connection Ξ is $\mathcal{G}$-invariant. We assume that $\mathcal{U}\to\Gamma(E)$ is a topologically trivial bundle and hence admits global sections. To any section S of $\mathcal{U}$ we associate a group cocycle α^S and a Lie algebra cocycle $\mathfrak{a}^S$. In this way the different expressions of the cocycle α and the integrated anomaly $\mathfrak{a}$ obtained from perturbation theory correspond to different sections of $\mathcal{U}\to\Gamma(E)$. Furthermore, S determines a trivialization of $\mathcal{U}\to\Gamma(E)$, and in this trivialization any other section is determined by a function of the form $\exp(2\pi i\Lambda)$. The condition (*) for topological anomaly cancellation is