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Chern–Simons deformation of vortices on compact domains

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ABSTRACT

Existence of Maxwell–Chern–Simons–Higgs (MCSH) vortices in a hermitian line bundle L over a general compact Riemann surface Σ is proved by a continuation method. The solutions are proved to be smooth both spatially and as functions of the Chern–Simons deformation parameter κ, and exist for all $|\kappa| < \kappa_*$, where κ_* depends, in principle, on the geometry of Σ, the degree n of L, which may be interpreted as the vortex number, and the vortex positions. A simple upper bound on κ_*, depending only on n and the volume of Σ, is found. Further, it is proved that a positive *lower* bound on κ_*, depending on Σ and n, but independent of vortex positions, exists. A detailed numerical study of rotationally equivariant vortices on round two-spheres is performed. We find that κ_* in general does depend on vortex positions, and, for fixed n and radius, tends to be larger the more evenly vortices are distributed between the North and South poles. A generalization of the MCSH model to compact Kähler domains Σ of complex dimension $k \geq 1$ is formulated. The Chern–Simons term is replaced by the integral over spacetime of $A \wedge F \wedge \omega^{k-1}$, where ω is the Kähler form on Σ. A topological lower bound on energy is found, attained by solutions of a deformed version of the usual vortex equations on Σ. Existence, uniqueness and smoothness of vortex solutions of these generalized equations is proved, for $|\kappa| < \kappa_*$, and an upper bound on κ_* depending only on the Kähler class of Σ and the first Chern class of L is obtained.

1. Introduction

Vortices are the simplest class of topological solitons occurring in gauge theory. Being simple, they are useful prototypes for more complicated, higher-dimensional solitons (monopoles, instantons, calorons), as well as having interesting applications in their own right, in condensed matter physics and cosmology. They arise in the abelian Higgs model, a $(2+1)$-dimensional field theory consisting of a complex scalar field φ (the Higgs field) minimally coupled to a $U(1)$ gauge field A, obeying Maxwell electrodynamics. For a particular choice of the Higgs self-interaction potential, this theory exhibits the mathematically interesting property of "self duality": there is a topological lower bound on energy which is attained by solutions of a coupled system of first order PDEs. The space of gauge equivalence classes of solutions of this system, in a fixed topological class, is a finite dimensional smooth manifold, the so called n-vortex moduli space M_n, which inherits a canonical Kähler structure. One may identify M_n with the space of unordered n-tuples of marked points in physical space, these being the points at which the Higgs field vanishes. This is true whether physical space is $\mathbb{R}^2$ [1] or a compact Riemann surface [2,3]. There is a well-developed formalism for extracting the low energy dynamics of vortices from the geometry of M_n, originally developed by Manton, see [4] for a thorough review.

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Many elaborations on the basic abelian Higgs model preserving a self-duality structure are possible. (One can, for example, allow both physical space and the target space of the Higgs field to be Kähler manifolds, and the gauge group to be any Lie group with a Hamiltonian and isometric action on target space.) From a physical standpoint, perhaps the most interesting elaboration is the inclusion of a Chern–Simons term in the theory. This converts the vortices into dyons, that is, particles carrying both magnetic flux and electric charge, and allows the possibility of exotic exchange statistics once the theory is quantized. There are two ways to introduce a Chern–Simons term into the theory while keeping Lorentz covariance and a self-duality structure. In one [5] the Maxwell term for A is directly replaced by the Chern–Simons term, and the usual quartic Higgs potential is replaced by a certain sextic potential. This Chern–Simons–Higgs (CSH) model has been quite thoroughly studied but, even so, the existence theory for vortices is less well developed than for standard vortices. It is known that an n-vortex exists for each choice of n points in physical space Σ if $\Sigma = \mathbb{R}^2$ [6, pp. 164-177] or a flat torus [7]. Once the model is put on a compact domain, the coupling constant in front of the Chern–Simons term, usually denoted κ, becomes a nontrivial parameter (on $\mathbb{R}^2$ it can be scaled away). It is proved in [7] that for each set D of n marked points on a torus, there exists $\kappa_*(D) > 0$, depending on D, such that, for all $\kappa \in (0, \kappa_*(D))$ there is a vortex solution with $\varphi^{-1}(0) = D$, and that $\kappa_*(D)$ is finite for all D (that is, for large enough κ, no n-vortex exists). Existence of vortices on compact Riemann surfaces of higher genus has not been established, and there does not appear to be a quick and simple resolution for this. In particular, a direct application of Bradlow's approach [2] to this vortex system is unhelpful because the higher nonlinearity of the sextic Higgs potential produces an elliptic PDE with analytically difficult nonlinear terms. Even recent studies of this type of vortex, dealing with the generalization to nonabelian gauge groups, restrict themselves to the case of flat tori [8].

In this paper, we address the vortex existence question in the second, rather less well-studied Chern–Simons vortex system, making progress on arbitrary compact domains. This model, originally due to Lee, Lee and Min [9] is a one-parameter *deformation* of the basic abelian Higgs model, the deformation parameter being the Chern–Simons coupling κ. The model keeps the usual Maxwell term for A, but adds (κ times) the Chern–Simons term, and couples the Higgs field φ to a new neutral scalar field N via a κ-dependent, but still quartic, interaction potential. We shall refer to it as the Maxwell–Chern–Simons–Higgs (MCSH) model. Following Bradlow [2], one can, for a fixed set of vortex positions, formulate the vortex equations as a coupled *pair* of semilinear second order elliptic PDEs, for $|\varphi|$ and N. Proving existence of solutions of systems of semilinear PDEs is, in general, a much more difficult problem than for a single PDE. Considerable progress has been made, in the case $\Sigma = T^2$ by Ricciardi and Tarantello [10]. By a thorough analysis of the coupled system, they establish that, for each set D of vortex positions, and for all $\kappa > 0$ sufficiently small, there are in fact at least *two* inequivalent vortex solutions with $\varphi^{-1}(0) = D$. They also find a global upper bound on $\kappa_*(D)$, depending only on the volume of Σ, and prove that, in two different limits, vortex solutions of the MCSH model converge to solutions of both the CSH and original abelian Higgs models.

It is plausible that the methods of [10] should extend to arbitrary compact Riemann surfaces Σ (and, indeed, existence results for MCSH vortices on general Σ are sometimes asserted as folk theorems on this basis [11]). That is not, however, the aim of the current paper. Rather, we will directly exploit the deformation character of the MCSH system to give a much simpler existence (and local uniqueness) proof of those vortices which continue smoothly to $\kappa = 0$. The idea is that, at $\kappa = 0$, we know a unique solution exists for each choice of D, namely the standard abelian Higgs vortex augmented by $N \equiv 0$. An Implicit Function Theorem argument then allows us to deduce that, for each D, there is $\kappa_*(D) > 0$ such that, for all $\kappa \in (-\kappa_*(D), \kappa_*(D))$ there is a locally unique vortex solution with $\varphi^{-1}(0) = D$. This solution is smooth, and depends smoothly on the deformation parameter κ. We also prove the existence of a positive lower bound κ_{**} on $\kappa_*(D)$, depending on n and Σ, but independent of D. Hence, for all position sets D of size n, locally unique vortices with $\varphi^{-1}(0) = D$ exist for all $-\kappa_{**} < \kappa < \kappa_{**}$. Loosely, this shows that, for sufficiently small κ, the entire moduli space of n-vortices M_n survives the Chern–Simons deformation, a key underlying assumption of the various proposals for moduli space approximations to low energy vortex dynamics in this model [12–14]. As far as we are aware, this is the first time existence of a global *lower* bound on κ_* has been established, and the smooth continuation viewpoint is crucial to our argument. In comparison with [10], we obtain more refined information (existence of *smooth curves* of solutions parametrized by κ, and a lower bound on κ_*) in more general geometries, but only for one type of vortex: those continuously connected to ordinary abelian Higgs vortices. Our argument is also considerably simpler, using only basic facts from functional analysis.

We will also find a global *upper* bound on $\kappa_*(D)$, independent of D. In the case $\Sigma = T^2$, this is larger (hence worse) than a bound obtained in [10], but, again, the proof is much simpler. Neither bound is expected to be sharp. Our bound may be thought of as the MCSH analogue of the Bradlow bound for existence of undeformed vortices [2], which states that n-vortices cannot exist if the volume of Σ is less than $4\pi n$. As we will see, Chern–Simons deformation makes this requirement more stringent: κ-deformed vortices cannot exist if $\mathrm{Vol}(\Sigma) < 4\pi n(1 + \kappa^2)$.

The question arises whether the maximal coupling $\kappa_*(D)$ at which vortices with $\varphi^{-1}(0) = D$ exist actually depends nontrivially on D. We will produce robust numerical evidence that it does, by studying the MCSH model on the round sphere (of radius $R \geq \sqrt{n}$) in the cases where D consists of the north pole, with multiplicity n_+, and the south pole with multiplicity $n_- = n - n_+$. In these cases, the Bogomol'nyi equations reduce, due to rotational invariance, to an ODE system, which we solve numerically via a shooting method. We find that, for given n and R, the maximal κ at which vortices exist depends on n_-, the number placed at the south pole. For example, $(n_+, n_-) = (1, 1)$ vortices have larger κ_* than $(n_+, n_-) = (2, 0)$ vortices. We also find that the solution curves, rather than disappearing at $\kappa = \kappa_*$, have a turning point in κ, continuing smoothly back towards $\kappa = 0$, approaching a singular limit in which the magnetic field becomes uniform, the Higgs field vanishes, and the neutral scalar field becomes uniform and diverges to $-\infty$. This indicates that the two distinct vortex solutions whose existence for each D (on T^2) and small $\kappa > 0$ was proved in [10], merge at $\kappa = \kappa_*(D)$, and can actually be considered as a single, connected solution branch.