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Circle actions on oriented manifolds with discrete fixed point sets and classification in dimension 4

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ABSTRACT

In this paper, we study a circle action on a compact oriented manifold with a discrete fixed point set. The fixed point data consists of the weights at the fixed points. We prove various results and properties of the action, in terms of the fixed point data. We show that the manifold can be described by a multigraph associated to it.

Specializing into the case of dimension 4, we classify the fixed point data. Moreover, we prove that there exist circle actions on oriented manifolds with these fixed point data. Finally, we show that a certain multigraph behaves like a manifold.

1. Introduction

In this paper, we study circle actions on oriented manifolds with discrete fixed point sets. Let the circle act on a compact oriented manifold M with a discrete fixed point set. At each fixed point, there are non-zero integers, called *weights* (also called *rotation numbers*). In this paper, we prove properties of the weights at the fixed points and derive results on the manifold. One of which we show is that we can associate a multigraph to M and the manifold can be described by the multigraph. Finally, we specialize into the case of dimension 4. In dimension 4, we classify the weights at the fixed points and prove the existence part. Moreover, we give a necessary and sufficient condition for a multigraph to be realized as a multigraph associated to a 4-dimensional oriented S^1-manifold.

Consider a circle action on a compact oriented manifold. Assume that the fixed point set is non-empty and finite. For the classification of such an action, one may want to begin either with small numbers of fixed points, or with low dimensions. Note that having an isolated fixed point implies that the dimension of the manifold is even.

First, let us begin with small numbers of fixed points. If there is one fixed point, then the manifold must be a point. On the other hand, if there are two fixed points, then any even dimension is possible, as there is an example of a rotation of an even dimensional sphere with two fixed points. From the example, on any even dimension greater than two, we can have any even number of fixed points, since we can perform equivariant sum of rotations of even dimensional spheres. This makes a big difference with S^1-actions on other types of manifolds; for instance, an almost complex (and complex or symplectic) manifold M equipped with an S^1-action[1] having two fixed points must have either $\dim M = 2$ or $\dim M = 6$; for the classification results for other types of manifolds, see [1–3].

The situation is quite different when the number of fixed points is odd. If there is an odd number of fixed points, then the dimension of the manifold must be a multiple of four; see Corollary 2.7. Let us specialize into the case of three fixed points. Then the complex, quaternionic, and octonionic (Cayley) projective spaces ($\mathbb{CP}^2$, $\mathbb{HP}^2$, and $\mathbb{OP}^2$) of dimension 2 admit circle actions with three fixed points, which have real dimensions 4, 8, and 16, respectively. On the other hand, to the author's

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1 Throughout the paper, if the circle acts on an almost complex, complex, or symplectic manifold, we assume that the action preserves the almost complex, complex, or symplectic structure, respectively.

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knowledge, it is not known if in dimensions other than 4, 8, and 16, there exists a manifold with three fixed points. Similar to the case of two fixed points, if we assume an almost complex structure on a manifold, three fixed points can only happen in dimension 4 [1]. Note that among the spaces above, only $\mathbb{CP}^2$ admits an almost complex structure (and hence complex or symplectic structures). For the classification results for other types of manifolds, see [4].

Second, let us begin with low dimensions. In dimension two, the classification is rather trivial. Among compact oriented surfaces, only the 2-sphere S^2 and the 2-torus $\mathbb{T}^2$ admit non-trivial circle actions. Any circle action on S^2 has two fixed points and any circle action on $\mathbb{T}^2$ is fixed point free; see Lemma 2.12.

We discuss classification results in dimension four. Before we discuss our main result, let us discuss results for other types of manifolds, or related results. The classification of a holomorphic vector field on a complex surface is made by Carrell, Howard, and Kosniowski [5]. For a 4-dimensional Hamiltonian S^1-space, subsequent to the work by Ahara and Hattori [6] and Audin [7], Karshon classifies such a space up to equivariant symplectomorphism, in terms of a multigraph associated to M [8]. Note that in Section 4, we shall associate a multigraph to an oriented manifold M and our notion of multigraphs generalizes the multigraphs for 4-dimensional Hamiltonian S^1-spaces. A multigraph determines the weights at fixed points. In dimension 4, for a complex manifold or for a symplectic manifold, the weights at the fixed points determine the manifold uniquely. When the fixed point set are discrete, our main result generalizes the classification of weights at fixed points for complex manifolds and symplectic manifolds to oriented manifolds. However, for a manifold to be oriented is a very weak condition, and therefore uniqueness fails to hold for oriented manifolds, since we can perform equivariant sum of a manifold with another manifold that is fixed point free.

Somewhat related results are the classifications of a circle action on a 4-manifold, with different perspectives. For instance, a circle action on a homotopy 4-sphere [9–12] or on a simply connected 4-manifold [13,14] is considered. In addition, Fintushel classifies a 4-dimensional oriented S^1-manifold in terms of orbit data [15].

Third, there is another point of view that one considers for a circle action on an oriented manifold with a discrete fixed point set. One of them is the Petrie's conjecture, which asserts that if a homotopy $\mathbb{CP}^n$ admits a non-trivial S^1-action, then its total Pontryagin class is the same as that of $\mathbb{CP}^n$ [16]. In other words, the existence of a non-trivial S^1-action is enough to determine the characteristic class of such a manifold. The Petrie's conjecture is proved to hold in dimension up to 8 [17,18].

To state our classification result, we introduce a terminology. Let the circle act on a $2n$-dimensional compact oriented manifold M with a discrete fixed point set. Let p be a fixed point. Then the tangent space at p decomposes into n two-dimensional irreducible S^1-equivariant real vector spaces

$$T_pM = \bigoplus_{i=1}^{n} L_i.$$

Each L_i is isomorphic to a one-dimensional S^1-equivariant complex space on which the action is given as multiplication by $g^{w_p^i}$, where $g \in S^1$ and w_p^i is a non-zero integer. The w_p^i are called **weights** at p. Though the sign of each weight is not well-defined, the sign of the product of the weights at p is well-defined. We orient each L_i so that every weight is positive. Let $\epsilon(p) = +1$ if the orientation given on $\bigoplus_{i=1}^{n} L_i$ this way agrees on the orientation on T_pM and $\epsilon(p) = -1$ otherwise. Let us call it the **sign** of p. Denote the fixed point data at p by $\Sigma_p = \{\epsilon(p), w_p^1, \dots, w_p^n\}$. By the fixed point data Σ_M of M, we mean a collection $\cup_{p \in M^{S^1}} \Sigma_p$ of the fixed point data at each fixed point p. To avoid possible confusion with weights, when we write the sign at p inside Σ_p, we shall only write the sign of $\epsilon(p)$ and omit 1.

We give an example. Let the circle act on S^{2n} by

$$g \cdot (z_1, \dots, z_n, x) = (g^{a_1} z_1, \dots, g^{a_n} z_n, x),$$

for any $g \in S^1 \subset \mathbb{C}$, where z_i are complex numbers and x is a real number such that $\sum_{i=1}^n |z_i|^2 + x^2 = 1$, and a_i are positive integers for $1 \leq i \leq n$. The action has two fixed points, $p = (0, \dots, 0, 1)$ and $q = (0, \dots, 0, -1)$. Near p, the action is described as $g \cdot (z_1, \dots, z_n) = (g^{a_1} z_1, \dots, g^{a_n} z_n)$. Therefore, the weights at p are $\{a_1, \dots, a_n\}$. Similarly, the weights at q are $\{a_1, \dots, a_n\}$. It is not hard to see that $\epsilon(p) = -\epsilon(q)$. The fixed point data of the circle action on S^{2n} is therefore $\{+, a_1, \dots, a_n\} \cup \{-, a_1, \dots, a_n\}$.

With the notion of weights, consider a circle action on a 4-dimensional compact oriented manifold M and assume that the fixed point set is discrete. As we have seen, the classification of the fixed point data for oriented manifolds is in general harder than complex manifolds or symplectic manifolds. To the author's knowledge, the fixed point data of an S^1-action on an oriented 4-manifold is known only if the number of fixed points is at most three; see [19] for the case of three fixed points. In this paper, we completely determine the fixed point data of M with an arbitrary number of fixed points. Note that given a circle action on a manifold, we can always make the action effective by quotienting out by the subgroup $\mathbb{Z}_k$ that acts trivially. This amounts to dividing all the weights by k. We prove that for a circle action on a 4-dimensional oriented manifold with a discrete fixed point set, the fixed point data of the manifold can be achieved by simple combinatorics. A combinatorial format of the main result can be stated as follows.

Theorem 1.1. *Let the circle act effectively on a 4-dimensional compact oriented manifold M with a discrete fixed point set. Then the fixed point data of M can be achieved in the following way: begin with the empty set, and apply a combination of the following steps.*

(1) *Add $\{+, a, b\}$ and $\{-, a, b\}$, where a and b are relatively prime positive integers.*