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A note on biharmonic functions on the Thurston geometries

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ABSTRACT

We construct new explicit proper biharmonic functions on the 3-dimensional Thurston geometries **Sol**, **Nil**, $\widetilde{\mathbf{SL}}_2(\mathbb{R})$, $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$.

1. Introduction

The biharmonic equation is a fourth order partial differential equation which arises in areas of continuum mechanics, including elasticity theory and the solution of Stokes flows. The literature on biharmonic functions is vast, but usually the domains are either surfaces or open subsets of flat Euclidean space $\mathbb{R}^n$.

Recently, new explicit local biharmonic functions were constructed on the classical compact simple Lie groups $\mathbf{SU}(n)$, $\mathbf{SO}(n)$ and $\mathbf{Sp}(n)$. This gives local solutions on the 3-dimensional round sphere $\mathbb{S}^3 \cong \mathbf{SU}(2)$ and the standard hyperbolic space $\mathbb{H}^3$ via a general duality principle. For this see the papers [1] and [2].

The classical Riemannian manifolds $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$ of constant curvature are all on Thurston's celebrated list of 3-dimensional model geometries, see [3,4] and [5]. The aim of this paper is to extend the investigation of biharmonic functions to the other members on Thurston's list i.e. **Sol**, **Nil**, $\widetilde{\mathbf{SL}}_2(\mathbb{R})$, $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$. In all these cases we construct new explicit solutions to the corresponding fourth order biharmonic equation.

Our methods can also be used to manufacture proper r-harmonic solutions for $r > 2$, see Definition 2.1. In this study we have chosen to mainly focus on the case when $r = 2$ because of its physical relevance. The results are formulated such that the solutions are globally defined but clearly the same constructions hold even locally. This is particularly important for the holomorphic functions in use.

2. Proper r-harmonic functions

Let (M, g) be a smooth m-dimensional manifold equipped with a Riemannian metric g. We complexify the tangent bundle TM of M to $T^{\mathbb{C}}M$ and extend the metric g to a complex-bilinear form on $T^{\mathbb{C}}M$. Then the gradient ∇f of a complex-valued

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function $f : (M, g) \to \mathbb{C}$ is a section of $T^{\mathbb{C}}M$. In this situation, the well-known *linear* Laplace–Beltrami operator Δ on (M, g) acts locally on f as follows

$$\Delta(f) = \operatorname{div}(\nabla f) = \sum_{i,j=1}^{m} \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_j} \left(g^{ij} \sqrt{|g|} \frac{\partial f}{\partial x_i} \right).$$

For two complex-valued functions $f, h : (M, g) \to \mathbb{C}$ we have the following well-known relation

$$\Delta(f \cdot h) = \Delta(f) \cdot h + 2 \cdot \kappa(f, h) + f \cdot \Delta(h), \qquad (2.1)$$

where the *conformality* operator κ is given by $\kappa(f, h) = g(\nabla f, \nabla h)$. Locally this acts by

$$\kappa(f, h) = \sum_{i,j=1}^{m} g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial h}{\partial x_j}.$$

Definition 2.1. For a positive integer r, the iterated Laplace–Beltrami operator Δ^r is given by

$$\Delta^0(f) = f \text{ and } \Delta^r(f) = \Delta(\Delta^{(r-1)}(f)).$$

We say that a complex-valued function $f : (M, g) \to \mathbb{C}$ is

(a) *r-harmonic* if $\Delta^r(f) = 0$, and
(b) *proper r-harmonic* if $\Delta^r(f) = 0$ and $\Delta^{(r-1)}(f)$ does not vanish identically.

It should be noted that the *harmonic* functions are exactly r-harmonic for $r = 1$ and the *biharmonic* functions are the 2-harmonic ones. In some texts, the r-harmonic functions are also called *polyharmonic* of order r.

3. The model geometry Sol

The model space **Sol** on Thurston's list can be seen as the 3-dimensional solvable Lie subgroup

$$\mathbf{Sol} = \{ \begin{bmatrix} e^t & 0 & x \\ 0 & e^{-t} & y \\ 0 & 0 & 1 \end{bmatrix} \mid x, y, t \in \mathbb{R} \}$$

of $\mathbf{SL}_3(\mathbb{R})$. The metric on **Sol** is determined by the orthonormal basis $\{X, Y, T\}$ of its Lie algebra $\mathfrak{sol}$ given by

$$X = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In the global coordinates (x, y, t) on **Sol** this takes the following well-known form

$$ds^2 = e^{2t} dx^2 + e^{-2t} dy^2 + dt^2.$$

It is easily seen that the corresponding Laplace–Beltrami operator Δ and the conformality operator κ satisfy

$$\Delta(f) = e^{-2t} \frac{\partial^2 f}{\partial x^2} + e^{2t} \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial t^2},$$

and

$$\kappa(f, h) = e^{-2t} \frac{\partial f}{\partial x} \frac{\partial h}{\partial x} + e^{2t} \frac{\partial f}{\partial y} \frac{\partial h}{\partial y} + \frac{\partial f}{\partial t} \frac{\partial h}{\partial t},$$

respectively.

We now present two new families of globally defined complex-valued proper biharmonic functions on the model geometry **Sol**.

Example 3.1. For non-zero elements $a, b \in \mathbb{C}^4$ let the complex-valued functions $f_1, f_2 : \mathbf{Sol} \to \mathbb{C}$ be defined by

$$f_1(x, y, t) = (a_1 + a_2 x + a_3 y + a_4 xy)$$

and

$$f_2(x, y, t) = (b_1 + b_2 x + b_3 y + b_4 xy).$$

Then a simple calculation shows that the Laplace–Beltrami operator satisfies

$$\Delta(f_1) = \Delta(f_2) = 0,$$