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Gauge transformations for categorical bundles

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ABSTRACT

A gauge transformation of categorical principal bundles arises from a functorial isomorphism between such bundles. We determine the geometric nature of such gauge transformations. For a twisted-product categorical principal bundle whose structure group is given by a pair of Lie groups G and H we show that a pair consisting of a traditional gauge transformation θ, given by a G-valued function, and an $L(H)$-valued 1-form A^H determine a categorical gauge transformation. More general gauge transformations are also studied.

1. Introduction

The purpose of this paper is to develop a counterpart of the classical gauge transformation in the setting of categorical bundles. Briefly put, a categorical bundle is a structure, formulated in the language of category theory, that encodes a classical principal bundle equipped with connection and some additional structure. (Here and always we use the terms 'classical principal bundle' to mean a principal bundle, in the usual sense from topology and differential geometry, as distinct from a categorical principal bundle.) Just as a classical principal bundle has a structure group, a categorical principal bundle involves two structure groups. Our framework for categorical bundles is motivated by the geometric and physical background and is distinct from more category-theory motivated frameworks.

A gauge transformation, in its most basic form, is given by a smooth function

$$\theta : U \to G,$$

where U is an open subset of a manifold and G is a Lie group that describes the symmetries of a system. In terms of principal bundles, the function θ corresponds to the bundle automorphism

$$U \times G \to U \times G : (b, g) \mapsto \big(b, \theta(b)g\big),$$

where we think of $U \times G$ as the product bundle over U. A connection form can, in this context, be described by a smooth 1-form A_1 on U with values in $L(G)$, the Lie algebra of G; the effect of the gauge transformation θ on A_1 is to transform it into

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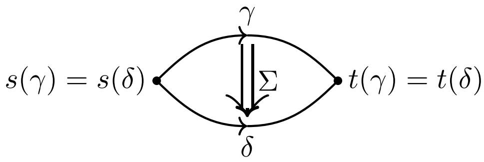

Fig. 1. A morphism of $\mathcal{P}_2(M)$.

the connection form A_2 given by

$$A_2 = \theta A_1 \theta^{-1} - (d\theta)\theta^{-1}. \tag{1.1}$$

In this paper we determine the counterpart of this for categorical principal bundles. Such a structure is given by a functor

$$\pi : \mathbf{P} \to \mathbf{M},$$

along with a categorical group $\mathbf{G}$ that acts functorially on the right on $\mathbf{P}$. We will explain these notions in Section 2, but for now let us note that a categorical group $\mathbf{G}$, when unraveled into non-categorical language, involves two Lie groups G and H, intertwined in a special structure.

A gauge transformation corresponds, in the categorical context, to a functorial bundle automorphism $\mathbf{P} \to \mathbf{P}$. We focus on the case where the categorical principal bundle is "trivial" in a certain special sense, with $\mathbf{P} = \mathbf{U} \times_\eta \mathbf{G}$ (this structure is described below in Eq. (3.8)), which contains geometric information beyond a simple product bundle structure. Theorem 4.1.1, which is one of our main results, provides an explicit determination of such a functor in the setting where the morphisms of $\mathbf{U}$ are given by paths on $U = \mathrm{Obj}(\mathbf{U})$, which is a manifold. Roughly stated, such a functor is specified by two 'gauge transformations': a G-valued function

$$\theta : U \to G$$

and an $L(H)$-valued 1-form Λ^H on U.

1.1. Other works and approaches

There is a considerable literature on category-theoretic approaches to gauge theories. A brief sample of this includes the many works of Baez et al. [1,2], Martins et al. [3–5], Parzygnat [6,7], Sati et al. [8], Schreiber et al. [9,10], Soncini and Zucchini [11], Waldorf [12–14], Wang [15–17].

Much of the literature mentioned above approaches the theory with a category-theoretic motivation. (The 'box category' structure used in Martins and Picken [5] is closer to our framework than is the standard 2-bundle theory.) The physics literature closest to our approach includes the works of Girelli and Pfeiffer [18,19]. Abbaspour and Wagemann [20] provide a brief comparison between some of the different approaches to higher gauge theory.

1.2. Comparison with other approaches

Our approach to categorical principal bundles, following the framework developed in our earlier papers [21,22], has a more geometric motivation and setting but uses category-theoretic structures to formulate the theory. We have developed this theory in several directions, including the construction of categorical bundles from local data [23], and in the study of twisted actions of categorical groups [24].

There are some basic differences between our framework and that of the 2-bundle approach. Fundamentally, our framework is a general one, that can be used to understand classical principal bundles as well as "higher" bundles over path spaces.

Let us first look at the situation for base spaces/categories. In the 2-category framework, the "higher path category" for the base manifold M is $\mathcal{P}_2(M)$, with objects corresponding to paths γ on M and morphisms $\Sigma : \gamma \to \delta$ running only between γ and δ that have a common source and a common target as shown in Fig. 1. In our framework, a higher morphism $\Gamma : \gamma_1 \to \gamma_2$ can run, in principle, between any two 'paths' γ_1 and γ_2 on M, as shown in Fig. 2. More generally, in our framework, $\mathrm{Obj}(\mathbf{M}_1) = \mathrm{Mor}(\mathbf{M})$, as we pass from a 'lower category' $\mathbf{M}$ to a higher category $\mathbf{M}_1$. Our approach is closer to the framework of double categories [25].

In our framework of *categorical principal bundles* there is a classical principal G-bundle that serves as 'object bundle', whereas such a structure does not directly appear in the 2-bundle approach. In other approaches the traditional cocycle defining a G-bundle is replaced by a weaker, functorial, notion, which also appears in our approach but in a different way [23].

Overall, our motivation is more differential geometric than category theoretic, and the central motivating examples, that of the decorated bundle (Section 2.14) and twisted-product bundles (Section 3), appear to be unique to our approach. At the bundle level, in the case of most interest in our framework, a morphism of the bundle category $\mathbf{P}$ is not simply a path on