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Geodesics on a Kerr–Newman–(anti-)de Sitter instanton

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A B S T R A C T

We study geodesics along a noncompact Kerr–Newman instanton, where the asymptotic geometry is either de Sitter or anti-de Sitter. We use first integrals for the Hamilton–Jacobi equation to characterize trajectories both near and away from horizons. We study the interaction of geodesics with special features of the metric, particularly regions of angular degeneracy or "theta horizons" in the de Sitter case. Finally, we characterize a number of stable equilibrium orbits.

1. Introduction

Much is known already regarding geodesics along Kerr spacetimes. Physically, these correspond to particles under the influence of a rotating black hole. Such geodesics have also been studied in the presence of charge and a nonzero cosmological constant (e.g. [1]). Less studied are geodesics along the corresponding gravitational instanton. In the Riemannian setting, there are no timelike curves that play the role of usual matter. That being said, geodesics in gravitational instantons do have physical interpretations: for instance, as the relative motion of certain monopoles. Notably, the Kaluza–Klein monopole can be embedded in a 5D Taub-NUT instanton, [2]. We will treat only a 4D instanton (of Kerr type) here, but the differential–geometric question of classifying its geodesics is still well-posed and may have applications to various scattering questions (monopole or otherwise) in instanton backgrounds. Generally speaking, Riemannian solutions to the Einstein equations have emerged as state transition probabilities in Euclidean quantum gravity [3–5] and are expected to encode quantum properties of their Lorentzian counterparts. Combining this with the general importance of the Kerr solution in the context of recent developments such as AdS/CFT and Kerr/CFT, understanding the geometric structure of the Riemannian analogue of Kerr is a priority.

Here, we consider a noncompact, incomplete Kerr–Newman–(anti-)de Sitter instanton. One can employ a periodic identification of the imaginary time coordinate to remove the singularity and create a compact version of the instanton, as suggested in [6,7] and then carried out and explored in [4,5]. This compact space will have a complete, everywhere positive-definite Einstein metric. We, on the other hand, obtain the instanton metric from a Wick rotation but do not perform the periodic identification. As such, the solution has degeneracies and signature changes. In particular, it is only Riemannian on certain submanifolds, and so one caveat is that the solution is a gravitational instanton only in a weaker sense.

Given that geodesics along the manifold in question possess fixed values of the first integrals that correspond usually to rest mass, energy, and angular momentum, we refer to them in any event as "particles" (even if they originate in the Riemannian part of the manifold, where they are spacelike). For both the de Sitter and anti-de Sitter geometries, we study the effect of the first integrals on the geodesic evolution, paying particular attention to their potential functions and to their sensitivity to the cosmological constant, the charge, and whether or not the singularity is slowly or rapidly rotating. In the

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Table 1.1
Block structure of AdS Δ_r roots.

$r \in (r_+, \infty)$	AdS Block
$r \in (r_-, r_+)$	Block I
$r \in (-\infty, r_-)$	Block II

Table 1.2
Block structure of dS Δ_r roots.

$r \in (r_{++}, \infty)$	dS_+ Block
$r \in (r_+, r_{++})$	Block I
$r \in (r_- r_+)$	Block II
$r \in (r_{--}, r_-)$	Block III
$r \in (-\infty, r_{--})$	dS_- Block

de Sitter situation, the metric degenerates in an interesting fashion – there are surfaces of angular degeneracy (as referred to in [8]) – and we study the trajectories of these particles near these degenerations. We also characterize a number of stable equilibrium orbits in this case.

1.1. Metric

Here, we consider the Kerr–Newman–(anti)-de Sitter, or KN(A)dS, instanton, which has underlying 4-manifold $M = \mathbb{R}^2 \times S^2$ and a metric g that solves the Riemannian Einstein–Maxwell field equations with positive (negative) cosmological constant. The field equations are

$$R_{\mu\nu} + \left(\Lambda - \frac{1}{2}R\right) g_{\mu\nu} = 2\left(F_{\mu\alpha}F^{\alpha}_{\nu} - \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g_{\mu\nu}\right)$$
$$dF = d \star F = 0,$$

where $g_{\mu\nu}$ is the metric, $R_{\mu\nu}$ and R are the Ricci tensor and scalar curvature respectively, Λ is the cosmological constant (a fixed real number), F is the Maxwell field, and $\star$ is the Hodge star. Choosing Λ positive results in a geometry that is asymptotically de Sitter while Λ negative results in an anti-de Sitter geometry.

After fixing Λ (and setting the magnetic charge parameter p to 0), there is a family of Lorentzian KN(A)dS metrics depending on 3 real numbers: M, the total mass of the instanton; a, the angular momentum per unit mass; and e, the charge per unit mass. That such solutions to the Einstein–Maxwell equations are parametrized by a, e, M is essentially the "No Hair Theorem" of general relativity. The KN(A)dS instanton solution is obtained from the corresponding spacetime solution in the standard way via a Wick transformation:

$$e \mapsto ie, \;\; a \mapsto ia, \;\; \text{and } t \mapsto it,$$

where $i = \sqrt{-1}$ (cf. [4,5] for instance). The Wick-rotated metric g then has the following line element in Boyer–Lindquist coordinates (r, t, θ, φ) on $\mathbb{R}^2 \times S^2$:

$$ds^2 = \frac{\Sigma}{\Delta_r}dr^2 + \frac{\Sigma}{\Delta_\theta}d\theta^2 + \frac{S^2}{\Xi^2\Sigma}\Delta_\theta(adt + (r^2 - a^2)d\varphi)^2 + \frac{\Delta_r}{\Xi^2\Sigma}(dt - aS^2d\varphi)^2,$$

in which we have functions

$$\Sigma = r^2 - a^2\cos^2\theta$$
$$\Delta_r = (r^2 - a^2)(1 - Lr^2) - 2Mr - e^2$$
$$\Delta_\theta = 1 - La^2\cos^2\theta$$
$$\Xi = 1 - La^2,$$

with $L = \frac{\Lambda}{3}$. We also note that the Maxwell potential under which g solves the field equations is

$$A = \frac{er}{\Sigma\Xi}(-dt + a\sin^2\theta d\varphi).$$

We partition the instanton in accordance with the roots of the Δ_r function. There are exactly two real roots for the anti-de Sitter instanton and four for its de Sitter counterpart (except for at extreme values of the parameters, as discussed in the Appendix). The partitions are defined in Tables 1.1 and 1.2.

Note that there will only exist one negative root for both instantons. This is r_- in the AdS instanton, and r_{--} in the dS instanton. The details of determining these root structures are deferred to the Appendix.

It is worth noting that, while we use the word "instanton" to describe the metric, this is only a gravitational instanton in an incomplete sense: we do not impose the ALE decay at infinity and the metric signature is not constant. (This is in contrast