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Index pairing with Alexander–Spanier cocycles

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To Alain Connes, with admiration and deep appreciation.

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ABSTRACT

We give a uniform construction of the higher indices of elliptic operators associated to Alexander–Spanier cocycles of either parity in terms of a pairing à la Connes between the K-theory and the cyclic cohomology of the algebra of complete symbols of pseudodifferential operators, implemented by means of a relative form of the Chern character in cyclic homology. While the formula for the lowest index of an elliptic operator D on a closed manifold M (which coincides with its Fredholm index) reproduces the Atiyah–Singer index theorem, our formula for the highest index of D (associated to a volume cocycle) yields an extension to arbitrary manifolds of any dimension of the Helton–Howe formula for the trace of multicommutators of classical Toeplitz operators on odd-dimensional spheres. In fact, the totality of higher analytic indices for an elliptic operator D amount to a representation of the Connes–Chern character of the K-homology cycle determined by D in terms of expressions which extrapolate the Helton–Howe formula below the dimension of M.

1. Introduction

The intuition that the Alexander–Spanier version of cohomology was an yet untapped resource for index theory has been one of the surprising insights that made the original list of topics in Alain Connes' master plan for noncommutative geometry [1, Introduction]. It has materialized in [2], where higher indices of elliptic operators associated to Alexander–Spanier cocycles were used to prove the Novikov conjecture for manifolds with word-hyperbolic fundamental group.

For an elliptic operator D between two vector bundles over a closed manifold M, the higher analytic index $\mathrm{Ind}_\phi D \in \mathbb{C}$ corresponding to an even-dimensional Alexander–Spanier cocycle ϕ on M was constructed essentially by folding the Alexander–Spanier cohomology of the manifold into cyclic cohomology for pseudodifferential operators, by exploiting the fact that the parametrices of D can be localized at will near the diagonal. The resulting number $\mathrm{Ind}_\phi D$ depends only on the cohomology class $[\phi] \in H^{ev}(M, \mathbb{C})$; in particular $\mathrm{Ind}_1 D$ coincides with the Fredholm index of D. After showing that these higher indices admit cohomological expressions akin to the Atiyah–Singer index formula [3], Connes and Moscovici proved a generalization of the Γ-index theorem (cf. Atiyah [4] and Singer [5]), which was instrumental in their proof [2] of the homotopy invariance of higher signatures.

In a different role, the pairing between Alexander–Spanier cohomology and the signature operator was used to produce local expressions for the rational Pontryagin classes of topological manifolds (quasiconformal in [6] and Lipschitz in [7]), and also for the Goresky–MacPherson $\mathcal{L}$-class of Witt spaces [8,9].

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In the aforementioned applications the odd case of the pairing, associating higher analytic indices of selfadjoint elliptic operators to odd-dimensional Alexander–Spanier cocycles, was handled in an indirect manner, essentially by reducing it via Bott suspension to the even case. The lack of a natural uniform definition irrespective of parity remained though a challenging conundrum, at least in the minds of the present authors. It is the purpose of this paper to provide a unified construction for higher analytic indices of either parity, based on recasting their definition within the conceptual framework of Connes' pairing between the K-theory and the cyclic cohomology of an algebra. Instead of the algebra of functions, the appropriate algebra for the task at hand is the algebra of complete symbols of pseudodifferential operators, over which the pairing is implemented by means of a relative form of the Chern character in cyclic homology.

Perhaps the quickest way to illustrate the flavor of the present picture for the higher index pairing (described in Section 2) is to mention its extreme cases. While $\mathrm{Ind}_1 D$ is just the Fredholm index of the elliptic operator D, the higher analytic index of D associated to a top-dimensional Alexander–Spanier (volume) cocycle on a manifold M has an expression reminiscent of the Helton–Howe formula [10, §7] for the trace of the top multicommutator of Toeplitz operators on an odd-dimensional sphere. As a matter of fact our formula represents a twofold extension of the Helton–Howe formula, to arbitrary manifolds and to any dimension, regardless of parity.

It was already noticed by Connes (cf. [1, Part I, §7]) that in the commutative case the total antisymmetrization of his Chern character in K-homology yields the Helton–Howe fundamental trace form. The cohomological formulas (proved in Section 3) for the higher analytic indices of an elliptic operator D on a closed manifold M show that, conversely, the full Connes–Chern character of the K-homology cycle determined by D can be recovered by extrapolating the Helton–Howe formula below the dimension of M.

2. Higher analytic indices

2.1. Higher Alexander–Spanier traces

Let M be a closed manifold. The (smooth) Alexander–Spanier complex of M is the quotient complex $\mathbf{C}^\bullet(M) = \{C^\bullet(M)/C^\bullet_0(M), \delta\}$, where $C^q(M) = C^\infty(M^{q+1})$, $C^q_0(M)$ consists of those $\phi \in C^q(M)$ which vanish in a neighborhood of the iterated diagonal $\Delta_{q+1}M = \{(x, \dots, x) \in M^{q+1}\}$, and

$$\delta\phi(x_0, x_1, \dots, x_{q+1}) = \sum (-1)^i \phi(x_0, \dots, \widehat{x_i}, \dots, x_{q+1}).$$

Its cohomology groups $H^\bullet_{AS}(M)$ yield the usual cohomology of M, alternatively computed from the Čech, the simplicial or the de Rham complex. The same cohomology is obtained from several variants of the Alexander–Spanier complex. We single out one of these variations which is particularly suited to the purposes of this paper. The specific complex consists of *decomposable* Alexander–Spanier cochains, i.e. finite sums of the form

$$\phi = \sum_\alpha f_0^\alpha \otimes f_1^\alpha \otimes \cdots \otimes f_q^\alpha, \qquad f_i^\alpha \in C^\infty(M),$$

which in addition are totally antisymmetric

$$\phi(x_{\nu(0)}, x_{\nu(1)}, \dots, x_{\nu(q)}) = \mathrm{sgn}(\nu)\phi(x_0, x_1, \dots, x_q), \qquad \forall\, \nu \in S_{q+1}.$$

The decomposable and totally antisymmetric cochains give rise to a quasi-isomorphic subcomplex $\mathbf{C}^\bullet_\wedge(M) = \{C^\bullet_\wedge(M)/C^\bullet_{\wedge,0}(M), \delta\}$ of the Alexander–Spanier complex $\mathbf{C}^\bullet(M)$.

On the other hand we consider the algebra of classical pseudodifferential operators $\Psi(M)$ on M, and the exact sequence

$$0 \to \Psi^{-\infty}(M) \to \Psi(M) \xrightarrow{\sigma} \mathcal{S}(M) \to 0, \tag{1}$$

where $\Psi^{-\infty}(M)$ is the ideal of smoothing operators and $\mathcal{S}(M)$ is the quotient algebra of complete symbols; σ denotes the complete symbol map.

To any cochain $\phi = \sum_i f_0^i \otimes f_1^i \otimes \cdots \otimes f_k^i \in C^k_\wedge(M)$ we associate the multilinear form Tr_ϕ on $\Psi^{-\infty}(M)$, defined by

$$\mathrm{Tr}_\phi(A_0, A_1, \dots, A_k) = \sum_i \mathrm{Tr}\left(A_0 f_0^i A_1 f_1^i \dots A_k f_k^i\right), \tag{2}$$

The assignment $C^\bullet_\wedge(M) \ni \phi \mapsto \mathrm{Tr}_\phi \in CC^\bullet(\Psi^{-\infty}(M))$ satisfies the coboundary identities

$$b\,\mathrm{Tr}_\phi = \mathrm{Tr}_{\delta\phi}, \quad B\,\mathrm{Tr}_\phi = 0\,; \tag{3}$$

the first one is tautological and the second follows from the total antisymmetry of ϕ.

To extend this assignment to the full algebra of pseudodifferential operators we use zeta-regularization of the operator trace. Fix an elliptic operator $R \in \Psi^1(M)$, invertible and positive. For $A \in \Psi^p(M)$ form

$$\zeta(s) = \mathrm{Tr}\, A R^{-s}, \qquad \Re(s) \gg 0. \tag{4}$$