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# Projective superspaces in practice

Sergio Luigi Cacciatori [a], Simone Noja [b,*]

[a] Dipartimento di Scienza e Alta Tecnologia, Università dell'Insubria, Via Valleggio 11, 22100 Como, Italy
[b] Dipartimento di Matematica, Università degli Studi di Milano, Via Saldini 50, 20133 Milano, Italy

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ABSTRACT

This paper is devoted to the study of supergeometry of complex projective superspaces $\mathbb{P}^{n|m}$. First, we provide formulas for the cohomology of invertible sheaves of the form $\mathcal{O}_{\mathbb{P}^{n|m}}(\ell)$, that are pullbacks of ordinary invertible sheaves on the reduced variety $\mathbb{P}^n$. Next, by studying the even Picard group $\mathrm{Pic}_0(\mathbb{P}^{n|m})$, classifying invertible sheaves of rank $1|0$, we show that the sheaves $\mathcal{O}_{\mathbb{P}^{n|m}}(\ell)$ are *not* the only invertible sheaves on $\mathbb{P}^{n|m}$, but there are also new genuinely supersymmetric invertible sheaves that are unipotent elements in the even Picard group. We study the $\Pi$-Picard group $\mathrm{Pic}_\Pi(\mathbb{P}^{n|m})$, classifying $\Pi$-invertible sheaves of rank $1|1$, proving that there are also non-split $\Pi$-invertible sheaves on supercurves $\mathbb{P}^{1|m}$. Further, we investigate infinitesimal automorphisms and first order deformations of $\mathbb{P}^{n|m}$, by studying the cohomology of the tangent sheaf using a supersymmetric generalisation of the Euler exact sequence. A special attention is paid to the meaningful case of supercurves $\mathbb{P}^{1|m}$ and of Calabi–Yau's $\mathbb{P}^{n|n+1}$. Last, with an eye to applications to physics, we show in full detail how to endow $\mathbb{P}^{1|2}$ with the structure of $\mathcal{N}=2$ super Riemann surface and we obtain its SUSY-preserving infinitesimal automorphisms from first principles, that prove to be the Lie superalgebra $\mathfrak{osp}(2|2)$. A particular effort has been devoted to keep the exposition as concrete and explicit as possible.



## 1. Introduction

The aim of this paper is to study the supergeometry of complex projective superspaces in some depth. Indeed, even if projective superspaces are considered well-understood supermanifolds – they are *split* supermanifolds and different realisations are known [1–4] – and they entered several well-known formal constructions in theoretical physics (see for example [5–7]), some of their geometric structures, constructions and properties have never been established on a solid basis and investigated in detail. In this paper we would like to fill this gap and give a complete and rigorous treatment of the subject.

After having reviewed the main definitions in the theory of supermanifolds and the various constructions of projective superspaces, we concentrate on invertible sheaves of rank $1|0$ – we call them *even invertible sheaves* – on $\mathbb{P}^{n|m}$. We carry out a detailed computation of Čech cohomology of the sheaves of the form $\mathcal{O}_{\mathbb{P}^{n|m}}(\ell)$, thus providing a (partial) supersymmetric analog of the celebrated *Bott formulas* for ordinary projective spaces.

Then, we study the *even Picard group* of projective superspaces, $\mathrm{Pic}_0(\mathbb{P}^{n|m})$, that classifies locally-free sheaves of rank $1|0$ over $\mathbb{P}^{n|m}$. In particular, we show that in the case of the supercurves $\mathbb{P}^{1|m}$ the even Picard group has a continuous part and we give the explicit form of its generators, proving that there exist genuinely supersymmetric invertible sheaves on $\mathbb{P}^{n|m}$ that

* Corresponding author.
*E-mail addresses:* sergio.cacciatori@uninsubria.it (S.L. Cacciatori), simone.noja@unimi.it (S. Noja).

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do not come from any ordinary invertible sheaves $\mathcal{O}_{\mathbb{P}^n}(\ell)$ on $\mathbb{P}^n$. These prove to be non-trivial geometric objects, indeed they have in general a non-trivial cohomology, as we show by means of an example.

Further, we show that the case of the supercurves $\mathbb{P}^{1|m}$ proves to be special also when looking at the *Π-invertible sheaves*, that is sheaves of rank 1|1 with a certain "exchange symmetry" between their even and odd part, called Π-symmetry. Indeed, when looking at Π-invertible sheaves on $\mathbb{P}^{1|m}$, one finds that for $m > 2$, beside the usual *split Π*-invertible sheaves of the kind $\mathcal{L} \oplus \Pi\mathcal{L}$, for $\mathcal{L}$ a locally-free sheaf of rank 1|0 on $\mathbb{P}^{1|m}$, one gets *non-split Π*-invertible sheaves as well, that cannot be presented in the split form.

Later, we study the cotangent sheaf on a generic, possibly non-projected, supermanifold, establishing the short exact sequences it fits into. We then specialise to the case of projected supermanifolds, proving that in this case the Berezinian sheaf – that plays a fundamental role in the theory of integration on supermanifolds – can be reconstructed by means of elements defined on the reduced variety, i.e. the canonical bundle of the reduced variety $\mathcal{M}_{red}$ itself and the determinant of the fermionic sheaf, that is actually a sheaf of locally-free $\mathcal{O}_{\mathcal{M}_{red}}$-modules. This allows to establish on a rigorous basis a result that has been already used, especially in theoretical physics: $\mathrm{Ber}(\mathbb{P}^{n|m}) \cong \mathcal{O}_{\mathbb{P}^{n|m}}(m-n-1)$. In particular, we call supermanifolds having trivial Berezinian sheaf – such as $\mathbb{P}^{n|n+1}$ for each $n \geq 1$ – *generalised Calabi–Yau supermanifolds* (henceforth Calabi–Yau supermanifolds, see [8] and [9] for some issues related to this definition in supergeometry), by similarity with the ordinary setting. This is so because the Berezinian sheaf somehow plays the role of canonical sheaf on ordinary manifolds: indeed, as in the ordinary setting one integrates sections of the canonical sheaf, in a supersymmetric setting one integrates sections of the Berezinian sheaf instead.

Next, using a supersymmetric generalisation of Euler exact sequence, we study the cohomology of the tangent sheaf of $\mathbb{P}^{n|m}$, which is related to the *infinitesimal automorphisms* and the *first order deformations* of $\mathbb{P}^{n|m}$. In this context, we find that supercurves over $\mathbb{P}^1$ yield again the richest scenario, allowing for many deformations as their odd dimension increases.

Then, after we have dealt with the case of supercurves, the example of the Calabi–Yau supermanifold $\mathbb{P}^{1|2}$ is examined. In particular, we show in full detail how to endow $\mathbb{P}^{1|2}$ with a structure of $\mathcal{N} = 2$ *super Riemann surface*. In this context, we show how to recover from first principles the $\mathcal{N} = 2$ SUSY-preserving automorphisms of $\mathbb{P}^{1|2}$ when structured as a $\mathcal{N} = 2$ super Riemann surface. These SUSY-preserving automorphisms prove to be isomorphic to the Lie superalgebra $\mathfrak{osp}(2|2)$, we give a physically relevant presentation of, by exhibiting a particularly meaningful system of generators and displaying their structure equations.

Next we briefly address how to give the structure of $\mathcal{N} = 2$ semi-rigid Riemann surface to $\mathbb{P}^{1|2}$ by means of a certain *twist*: this supermanifold is the basic ingredient in genus 0 *topological string theory*, a topic that we will address from a purely supergeometric point of view in a forthcoming paper. To conclude, we study the supergeometry of the so-called twistorial Calabi–Yau $\mathbb{P}^{3|4}$, that enters many constructions in theoretical physics. A supersymmetric generalisation of the exponential exact sequence – we call it *even* exponential exact sequence – , that is used throughout the paper is proved in the Appendix.

## 2. General definitions and notations

Before we start, we give some fundamental definitions in supergeometry and we fix some notations. For a more thorough treatment of the general theory of supermanifolds, see for example [3] or the first chapter of the recent [10].

First of all, on the algebraic side, we recall that a superalgebra is a $\mathbb{Z}_2$-graded algebra $A = A_0 \oplus A_1$, whose *even* elements commute and whose *odd* elements anti-commute.

The most basic example of complex supermanifold is the complex superspace $\mathbb{C}^{n|m} := (\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n} \otimes \bigwedge[\xi_1, \ldots, \xi_m])$, that is the ringed space whose underlying topological space is given by $\mathbb{C}^n$ endowed with the complex topology and whose structure sheaf is given by the sheaf of superalgebras obtained by tensoring the sheaf of holomorphic function on $\mathbb{C}^n$ with the Grassmann algebra generated by $m$ elements $\{\xi_1, \ldots, \xi_m\}$.

More generally, in the same fashion of the ordinary theory of complex manifolds, a *complex supermanifold* of dimension $n|m$ is a locally ringed space $(|\mathcal{M}|, \mathcal{O}_{\mathcal{M}})$, for $|\mathcal{M}|$ a topological space and $\mathcal{O}_{\mathcal{M}}$ a sheaf of superalgebras, which is *locally* isomorphic to $\mathbb{C}^{n|m}$ (in the $\mathbb{Z}_2$-graded sense). Clearly, a morphism of supermanifolds $\varphi : \mathcal{M} \to \mathcal{N}$ is therefore a morphism of locally ringed space (see [11]), that is a pair

$$(\phi, \phi^\sharp) : (|\mathcal{M}|, \mathcal{O}_{\mathcal{M}}) \longrightarrow (|\mathcal{N}|, \mathcal{O}_{\mathcal{N}}) \tag{1}$$

where $\phi : |\mathcal{M}| \to |\mathcal{N}|$ is a continuous map and $\phi^\sharp : \mathcal{O}_{\mathcal{N}} \to \phi_* \mathcal{O}_{\mathcal{M}}$ is a morphism of sheaves of superalgebras (thus respecting the $\mathbb{Z}_2$-grading). These are the kind of morphisms defining the local isomorphisms entering the definition of supermanifold given above.

To every supermanifold $\mathcal{M}$ is attached its reduced manifold $\mathcal{M}_{red}$, consisting, as a locally ringed space, of a pair $(|\mathcal{M}|, \mathcal{O}_{\mathcal{M}_{red}})$: this is an ordinary manifold, whose structure sheaf $\mathcal{O}_{\mathcal{M}_{red}}$ is given by the quotient $\mathcal{O}_{\mathcal{M}}/\mathcal{J}_{\mathcal{M}}$, where $\mathcal{J}_{\mathcal{M}}$ is the *nilpotent sheaf*, actually a sheaf of ideals in $\mathcal{O}_{\mathcal{M}}$ (looking at it as a usual classical scheme, $\mathcal{M}_{red}$, is a reduced scheme, while $\mathcal{M}$ is not). In other words, given any supermanifold $\mathcal{M}$, this corresponds to the existence of a *closed embedding* as follows:

$$\iota := (i, i^\sharp) : (|\mathcal{M}|, \mathcal{O}_{\mathcal{M}}/\mathcal{J}_{\mathcal{M}}) \to (|\mathcal{M}|, \mathcal{O}_{\mathcal{M}}). \tag{2}$$

Collectively, we will just denote it by $\iota : \mathcal{M}_{red} \to \mathcal{M}$. By the way, what really matters here, is that a supermanifold comes endowed with a sheaf morphism $i^\sharp : \mathcal{O}_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M}_{red}}$ (notice that $i : |\mathcal{M}| \to |\mathcal{M}|$ is the identity, for the underlying topological