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Queer Poisson brackets

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ABSTRACT

We give a method to construct Poisson brackets $\{\cdot, \cdot\}$ on Banach manifolds M, for which the value of $\{f, g\}$ at some point may depend on higher order derivatives of the smooth functions $f, g : M \to \mathbb{R}$ and not only on the first-order derivatives, as it is the case on all finite-dimensional manifolds. We discuss specific examples in this connection, as well as the impact on the earlier research on Poisson geometry of Banach manifolds. Those brackets are counterexamples to the claim that the Leibniz property for any Poisson bracket on a Banach manifold would imply the existence of a Poisson tensor for that bracket.

1. Introduction

The Poisson brackets in infinite-dimensional setting have played for a long time a significant role in various areas of mathematics including mechanics (both classical and quantum) and integrable systems theory (see e.g. [1–4]). However the rigorous approach to the notion of Poisson manifold in the context of Banach space is relatively recent (see [5] and also [6]). It is known that the Poisson brackets on infinite-dimensional manifolds lack some of the properties known from the finite-dimensional case. It was shown for instance in [5] that the existence of Hamiltonian vector fields requires an additional condition on the Poisson tensor in the case of manifolds modeled on a non-reflexive Banach space (i.e. a Banach space E that is not canonically isomorphic to its second dual $E \subsetneq E^{**}$, where E^* denotes the topological dual of a Banach space). Moreover on some manifolds, Poisson brackets need not be local although as far as we know a counterexample is not known yet, see a related discussion in [7].

The aim of this paper is to initiate the investigation of still another phenomenon that is specific to Poisson geometry on an infinite dimensional manifold M, namely the existence of Poisson brackets of higher order. That is, Leibniz property does not ensure that the bracket depends only on the first-order derivatives of functions. In particular, we construct Poisson brackets that are counterexamples to the statements given in the literature (see [5, Sect. 2] or subsequently [8, Sect. 1]), where it was claimed that the existence of a Poisson tensor Π follows from Leibniz property and skew symmetry of the Poisson bracket $\{\cdot, \cdot\}$, in particular for every $m \in M$ one could find a bounded bilinear functional $\Pi_m : T^*_m M \times T^*_m M \to \mathbb{R}$ satisfying

$$\{f, g\}(m) = \Pi_m(f'_m, g'_m)$$

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where $f'_m, g'_m \in T^*_m M$ are the differentials of $f, g \in C^\infty(M)$ at $m \in M$. A related fact in [3, Thm. 4.2.16] does hold, but we show that it is not applicable here (see Proposition 2).

Section 2 contains our main results, which show a general way to construct higher order Poisson brackets out of queer vector fields (a notion introduced in [9]). We conclude the paper with remarks on Hamiltonian vector fields associated to these general Poisson brackets, and we also suggest a definition of Banach Poisson manifolds that clarifies the one introduced in [5].

All Banach and Hilbert spaces considered in this paper are real. By manifold we will always mean a smooth real manifold modeled on a Banach space.

2. A construction of queer Poisson brackets

Our general construction of queer Poisson brackets (Theorem 3) needs some preparations on tangent vectors to Banach manifolds. There are two major approaches to tangent vectors, namely the kinematic one and the operational one. These approaches lead to the same notion for finite-dimensional manifolds, but it is well known that this is no longer the case in infinite dimensions. A kinematic tangent vector $v \in T_m M$ to a Banach manifold M at a point $m \in M$ is an equivalence class of curves passing through that point (for the precise definition see e.g., [3]). On the other hand, an operational tangent vector is defined as a derivation acting in the space of germs of functions (see [9]). Let $C^\infty_m(M)$ be the set of germs of all smooth functions at a point m. We denote by $L_k(T_m M; \mathbb{R})$ the Banach space of bounded k-linear functionals on $T_m M$ with values in $\mathbb{R}$ and let $f^{(k)}_m \in L_k(T_m M; \mathbb{R})$ be the kth differential at the point $m \in M$ of a germ or a function.

Definition 1. An **operational tangent vector** at a point $m \in M$ is a linear map $\delta : C^\infty_m(M) \to \mathbb{R}$ satisfying the Leibniz rule :

$$\delta(fg) = \delta f \; g(m) + f(m)\, \delta g. \tag{1}$$

For any open subset $U \subseteq M$ with $m \in U$ one has the map $C^\infty(U) \to C^\infty_m(M)$ that takes every function on U to its germ at m, and this leads to a canonical pull-back of δ to $C^\infty(U)$, also denoted by δ.

An **operational vector field** on M is a collection of maps $\delta_U : C^\infty(U) \to C^\infty(U)$ for each open set $U \subset M$, compatible with restrictions to open subsets and defining an operational tangent vector δ_m at every $m \in M$.

Definition 2. The operational tangent vector δ is of **order** n if it can be expressed in the form

$$\delta f = \sum_{k=1}^{n} \ell_k(f^{(k)}_m), \tag{2}$$

where $\ell_k : L_k(T_m M; \mathbb{R}) \to \mathbb{R}$ are continuous and linear, and ℓ_n does not vanish identically on the subspace of symmetric n-linear maps in $L_k(T_m M; \mathbb{R})$. Otherwise the order of δ is infinite. The operational tangent vectors of order at least 2 are called **queer**.

The operational vector field δ is of **order at most** n if one has (2) at all $m \in M$ for some family of smooth sections ℓ_k of the bundle $\bigsqcup_{m \in M} (L_k(T_m M; \mathbb{R}))^*$.

The existence of operational tangent vectors of order higher than three (or infinite) is an open problem as far as we know. The Leibniz rule (1) satisfied by δ implies certain algebraic conditions on functionals ℓ_k, see [9, 28.2]. Any kinematic tangent vector gives an operational tangent vector of order 1. All operational tangent vectors of order 1 are given by elements of $T^{**}M$.

We now show that there are queer operational tangent vectors on many Banach spaces, extending a construction from [9].

Proposition 1. *There are no operational tangent vectors of order two on the Banach space l^p of p-summable sequences for $2 < p < \infty$. If $1 \le p \le 2$, then there are non-trivial operational tangent vectors of order two on l^p.*

Proof. The proof of existence of operational tangent vectors of order two is inspired by [9, Rem. 28.8]. Namely, let E be a Banach space and consider the natural inclusion of $E^* \times E^*$ into $L_2(E; \mathbb{R})$ by

$$(f, g) \mapsto (f \otimes g : (v, w) \mapsto f(v)g(w)) .$$

In general (contrary to the finite-dimensional case) the linear span of its image may not be dense. A functional $\ell \in (L_2(E; \mathbb{R}))^*$ defines an operational tangent vector of order 2 at any $a \in E$ by

$$\delta_\ell f = \ell(f''_a) \tag{3}$$

if and only if it vanishes on the image of $E^* \times E^*$.

Hence the existence of an operational tangent vector of order 2 on l^p is equivalent to the existence of a nonzero continuous linear functional ℓ on $L_2(l^p; \mathbb{R}) \cong L(l^p; (l^p)^*)$ that vanishes on $(l^p)^* \times (l^p)^*$.

Every bounded linear map from l^p to $(l^p)^*$ is compact if $2 < p < \infty$, by Pitt's theorem [10, Thm. 4.23]. Moreover since all $(l^p)^*$ spaces have the approximation property, the closure of linear span of $(l^p)^* \times (l^p)^*$ coincides with the space of compact