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Rigidity of isometric immersions into the light cone

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ABSTRACT

In this paper, we show the rigidity of isometric immersions for a Riemannian manifold of dimension $n-1$ into the light cone of $n+1$ dimensional Minkowski, de Sitter and anti-de Sitter spacetimes for $n \geq 3$.

1. Introduction

Isometric embedding into Euclidean spaces is a classical subject in differential geometry. For examples, the Weyl problem (see [1,2]), Nash's embedding theorem (see [3]), etc. The book [4] by Han and Hong makes an excellent exposition of the subject, especially for isometric embeddings of surfaces into $\mathbb{R}^3$. Because of applications in General Relativity, isometric embeddings of Riemannian manifolds into Minkowski spacetime were also studied by experts. For example, the works [5–7] considered isometric embedding of codimension 1 into Minkowski spacetime. In [8–13], the authors defined quasi-local mass/energy by isometric embedding of a surface into $\mathbb{R}^{1,3}$. Previously, quasi-local mass was defined by isometric embedding into $\mathbb{R}^3$ in [14–16]. Note that isometric embedding of surfaces into $\mathbb{R}^{1,3}$ is of codimension 2 and hence lack of rigidity. To handle this problem, in [11,13], the authors consider critical isometric embeddings of certain energy functionals. There are other interesting proposals to handle this problem. In [9], Epp proposed to impose one more restriction $F = F_{\rm ref}$ on the isometric embedding to handle this problem, where F and $F_{\rm ref}$ are essentially the normal curvatures of the surface in the physical spacetime and the reference spacetime, respectively. On the other hand, in [8,10], the authors considered isometric embedding of surfaces into the light cone of $\mathbb{R}^{1,3}$, so that reduced the isometric embedding problem to the codimension 1 case. However, according to the authors's knowledge, rigidity of isometric embeddings in the two proposals has not been proved.

The existence of isometric embeddings into the light cone of $\mathbb{R}^{1,3}$ was first proved by Brinkmann [17] (see also [18–20]). Brinkmann showed that an $(n-1)$-dimensional Riemannian manifold (M, g) can be locally embedded into the light cone of $\mathbb{R}^{1,n}$ if and only if (M, g) is locally conformally flat. In fact, Brinkmann wrote down the isometric embedding explicitly by using the conformal factor. In this paper, we prove that the isometric embedding constructed by Brinkmann is essentially the unique isometric embedding into the light cone. Indeed, we also extend Brinkmann's result and rigidity of isometric embeddings to light cones of de Sitter and anti-de Sitter spacetimes.

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For the Minkowski spacetime $\mathbb{R}^{1,n}$, let $o^{(0)}$ be the origin, then the future light cone at $o^{(0)}$ (the set forms by future directed null geodesics starting at $o^{(0)}$) is

$$\mathcal{L}_+^{(0)} = \{(t, x) \in \mathbb{R}^{1,n} \mid -t^2 + \|x\|^2 = 0 \text{ and } t > 0\}. \tag{1.1}$$

The $n+1$ dimensional de Sitter spacetime dS_{n+1} can be defined as the hypersurface given by

$$-t^2 + x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1. \tag{1.2}$$

of $\mathbb{R}^{1,n+1}$ equipped with the induced metric. The Lorentz group $O(1, n+1)$ acts on dS_{n+1} transitively. Let $o^{(1)} = (0, 0, \ldots, 1)$. Then, the isotropy group at $o^{(1)}$ of the action is $O(1, n)$ by identifying $A \in O(1, n)$ with

$$A^{(1)} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in O(1, n+1). \tag{1.3}$$

Moreover, it is not hard to see that the future light cone at $o^{(1)}$ is

$$\mathcal{L}_+^{(1)} = \{(t, x) \in dS_{n+1} \mid x_{n+1} = 1, t > 0\}. \tag{1.4}$$

The $n+1$ dimensional anti-de Sitter spacetime AdS_{n+1} can be defined as the hypersurface

$$-t_1^2 - t_2^2 + x_1^2 + x_2^2 + \cdots + x_n^2 = -1 \tag{1.5}$$

of $\mathbb{R}^{2,n}$ equipped with the induced metric. The group $O(2, n)$ acts on AdS_{n+1} transitively. Let $o^{(-1)} = (1, 0, 0, \ldots, 0)$. Then, the isotropy group at $o^{(-1)}$ of the action is also $O(1, n)$ by identifying $A \in O(1, n)$ with

$$A^{(-1)} = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in O(2, n). \tag{1.6}$$

It is not hard to see that the future light cone of AdS_{n+1} at $o^{(-1)}$ is

$$\mathcal{L}_+^{(-1)} = \{(t, x) \in AdS_{n+1} \mid t_1 = 1, t_2 > 0\}. \tag{1.7}$$

The main result of this paper is as follows.

Theorem 1.1.

(1) *a connected Riemannian manifold (M, g) can be isometrically immersed into $\mathcal{L}_+^{(k)}$ for $k = 0, \pm 1$ if and only if (M, g) can be conformally immersed into $\mathbb{S}^{n-1}$ equipped with the standard metric;*

(2) *a closed connected Riemannian manifold (M, g) of dimension $n-1$ with $n \geq 3$ can be isometrically immersed into $\mathcal{L}_+^{(k)}$ for $k = 0, \pm 1$ if and only if (M, g) is conformally diffeomorphic to $\mathbb{S}^{n-1}$ equipped with the standard metric. Moreover, isometric immersion of (M, g) into $\mathcal{L}_+^{(k)}$ is rigid. More precisely, let $\varphi_1 : M \to \mathcal{L}_+^{(k)}$ and $\varphi_2 : M \to \mathcal{L}_+^{(k)}$ be two isometric immersions. Then, there is a unique $\tau \in O_+(1, n)$ such that $\varphi_2 = \tau^{(k)} \circ \varphi_1$ on M;*

(3) *isometric immersion of a connected $(n-1)$-dimensional Riemannian manifold (M, g) (not necessarily closed) into $\mathcal{L}_+^{(k)}$ for $k = 0, \pm 1$ is rigid with $n \geq 4$. More precisely, let $\varphi_1 : M \to \mathcal{L}_+^{(k)}$ and $\varphi_2 : M \to \mathcal{L}_+^{(k)}$ be two isometric immersions. Then, there is a unique $\tau \in O_+(1, n)$ such that $\varphi_2 = \tau^{(k)} \circ \varphi_1$ on M.*

Here $O_+(1, n)$ is the group of Lorentz transformations preserving time direction, $\tau^{(0)} = \tau$ and the definitions of $\tau^{(k)}$ for $k = 1$ and $k = -1$ are (1.3) and (1.6) respectively.

The result (1) in Theorem 1.1 can be viewed as an extension of Brinkmann's result since the standard metric on $\mathbb{S}^{n-1}$ is locally conformally flat. It was also obtained in [18] for Minkowski spacetime. The result (2) in Theorem 1.1 was also obtained in [18,19] for Minkowski spacetime.

It may be worth to note that (2) of Theorem 1.1 is not true for $n = 2$. For example, let

$$M = \{(2\cos\theta, 2\sin\theta) \mid \theta \in [0, 2\pi]\}$$

be a circle of radius 2. Let

$$\varphi_1(2\cos\theta, 2\sin\theta) = (2, 2\cos\theta, 2\sin\theta)$$

and

$$\varphi_2(2\cos\theta, 2\sin\theta) = (1, \cos(2\theta), \sin(2\theta)).$$

It is clear that φ_1 and φ_2 are both isometric immersions of M into the light cone of $\mathbb{R}^{1,2}$. However, there is no Lorentz transformation τ such that $\varphi_2 = \tau \circ \varphi_1$. Our result indicates that this kind of phenomenon never happens for $n \geq 3$.

Moreover, the result (3) of Theorem 1.1 is not true for $n = 3$. For example, let M be the unit disk of $\mathbb{C}$. Let

$$\overline{\varphi}_1 : M \to \mathbb{C} \cup \{\infty\} = \mathbb{S}^2$$