Accepted Manuscript

Four-dimensional pseudo-Riemannian g.o. spaces and manifolds

Giovanni Calvaruso, Amirhesam Zaeim

PII: S0393-0440(18)30173-6
DOI: https://doi.org/10.1016/j.geomphys.2018.03.018
Reference: GEOPHY 3187

To appear in: *Journal of Geometry and Physics*

Received date : 9 March 2017
Revised date : 21 March 2018
Accepted date : 24 March 2018

FOUR-DIMENSIONAL PSEUDO-RIEMANNIAN G.O. SPACES AND MANIFOLDS

GIOVANNI CALVARUSO AND AMIRHESAM ZAEIM

ABSTRACT. A g.o. manifold is a homogeneous pseudo-Riemannian manifold whose geodesics are all homogeneous, that is, they are orbits of a one-parameter group of isometries. A g.o. space is a realization of a homogeneous pseudo-Riemannian manifold (M, g) as a coset space $M = G/H$, such that all the geodesics are homogeneous.

We prove that apart form the already classified non-reductive examples [Calvaruso, Fino and Zaeim, 2015], any four-dimensional pseudo-Riemannian g.o. manifold is naturally reductive. To obtain this result, we shall also provide a complete description up to isometries of four-dimensional pseudo-Riemannian g.o. spaces, and show explicit realizations of the four-dimensional pseudo-Riemannian naturally reductive spaces classified in [Batat, Castrillon-Lopez and Rosado, 2015].

1. Introduction and background information

Let (M, g) be a homogeneous pseudo-Riemannian manifold. Denoted by $G \subset I_0(M, g)$ a connected Lie group of isometries acting transitively on M, the manifold can be identified with the pseudo-Riemannian homogeneous space $(G/H, g)$, where H denotes the isotropy group of the origin $P_0 \in M$.

A geodesic γ through $P_0 \in M = G/H$ is called *homogeneous* if it is the orbit of a one-parameter subgroup. In general, the group G is not unique. Clearly, if γ is homogeneous with respect to some isometry group G, then it is also homogeneous with respect to the maximal connected group of isometries, but not conversely.

A homogeneous pseudo-Riemannian manifold (M, g) is said to be *reductive* if $M = G/H$ and the Lie algebra $\mathfrak{g}$ of G can be decomposed into a direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is an $\mathrm{Ad}(H)$-invariant subspace of $\mathfrak{g}$. It is well known that when H is connected, this condition is equivalent to the algebraic condition $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$. With regard to the existence of a reductive decomposition, a fundamental difference arises between homogeneous Riemannian and pseudo-Riemannian manifolds. In fact, any homogeneous Riemannian manifold is reductive, but in dimension four and higher, there exist homogeneous pseudo-Riemannian manifolds which do not admit any reductive decomposition. Four-dimensional homogeneous non-reductive pseudo-Riemannian manifolds, both Lorentzian and of neutral signature, were classified in [23], and an explicit description of their invariant homogeneous metrics was obtained in [13].

2000 *Mathematics Subject Classification.* 53C10, 53C30, 53C50.

Key words and phrases. Homogeneous geodesics, g.o. spaces, g.o. manifolds, naturally reductive spaces.

First author partially supported by funds of GNSAGA and MIUR (within PRIN). Second author partially supported by funds of the University of Payame Noor.