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Isometry classes of simply connected nilmanifolds

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ABSTRACT

We classify the isometry equivalence classes and determine the isometry groups of connected and simply connected Riemannian nilmanifolds on filiform Lie groups of arbitrary dimension and on five dimensional nilpotent Lie groups of nilpotency class > 2. To achieve this classification we prove that up to one exceptional class the five dimensional non two-step nilmanifolds and the filiform nilmanifolds have isometry groups of the same (minimal) dimension as the nilmanifold. We give a detailed description of the exceptional case.

1. Introduction

A connected Riemannian manifold M is said to be a *Riemannian nilmanifold* if its group of isometries contains a nilpotent Lie subgroup acting transitively on this manifold. E. Wilson proved in [1] that there is a unique nilpotent Lie subgroup N of the group of isometries acting simply transitively on M and hence the Riemannian nilmanifold M can be identified with the nilpotent Lie group N endowed with a left-invariant metric $\langle .,.\rangle_N$. Moreover, the group $\mathcal{I}(N)$ of all isometries of $(N, \langle .,.\rangle_N)$ is isomorphic to the semi-direct product $N \rtimes \mathcal{OA}(\mathfrak{n})$, where $\mathcal{OA}(\mathfrak{n})$ is the group of orthogonal automorphisms of the Lie algebra $\mathfrak{n}$ of N with respect to the inner product induced on $\mathfrak{n}$ by the left-invariant metric $\langle .,.\rangle_N$. It follows from this observation that the isometry equivalence classes of connected and simply connected nilmanifolds and their isometry groups can be determined by the investigation of the classes of isometrically isomorphic metric Lie algebras, i.e. Lie algebras equipped with an inner product. This procedure was applied by J. Lauret in [2] to nilpotent Lie groups of dimension 3, 4 and to a 5-dimensional two-step nilpotent group, Sz. Homolya and O. Kowalski described in [3] the isometry equivalence classes and isometry groups of all 5-dimensional simply connected two-step nilpotent Riemannian nilmanifolds. Moreover, S. Console, A. Fino, E. Samiou determined in [4] the isometry equivalence classes and isometry groups of all 6-dimensional simply connected two-step nilpotent Riemannian nilmanifolds.

Riemannian nilmanifolds of higher dimension have a rich geometry with many open questions. Among nilmanifolds the class of higher nilpotency class, particularly the filiform manifolds have a relatively rigid structure, the papers [5] of M. M. Kerr and T. L. Payne, [6,7] of G. Cairns, A. Hinić Galić and Yu. Nikolayevsky have been devoted to the investigation of the geometry of filiform Riemannian nilmanifolds.

The aim of our paper is to investigate isometry equivalence classes and isometry groups of nilmanifolds on filiform Lie groups of arbitrary dimension and to extend the classification process of nilmanifolds to 5-dimensional nilpotent groups

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of nilpotency class greater than two. It turns out that the isometry groups of the investigated manifolds have minimal dimension, with the exception of one case, we give a detailed analysis of this nilmanifold.

The paper is organized as follows. In Section 2 we collect some basic definitions and notations and formulate the steps of our classification procedure. Section 3 is devoted to the study of filiform metric Lie algebras and of the isometry groups of the corresponding nilmanifolds. In Section 3.1 we introduce the notion of framed metric Lie algebras and show that filiform metric Lie algebras are framed, which has consequences on the structure of the connected component of the isometry group of the corresponding nilmanifolds. Section 3.2 deals with the classification of standard filiform metric Lie algebras and of the isometry groups of the corresponding nilmanifolds. The results are used to describe the 4- and 5-dimensional cases in detail. In Section 3.3 we study the non-standard filiform metric Lie algebra of smallest dimension 5. In Section 4 we complete the classification of 5-dimensional nilpotent metric Lie algebras and the corresponding isometry groups with the investigation of the two 3-step nilpotent metric Lie algebras. The such metric Lie algebras with one-dimensional center are framed metric algebras, hence we can apply in Section 4.1 our method of classification to this case. In contrast to the previous discussion there is a subclass of 5-dimensional 3-step nilpotent metric Lie algebras with two-dimensional center which does not have framing and hence the dimension of the isometry group of the corresponding nilmanifold is greater than 5. Section 4.2 is devoted to the detailed description of these metric Lie algebras and the corresponding nilmanifolds.

2. Preliminaries

In this paper we investigate on the one hand *filiform Lie algebras*. Denoting the *lower central series* of a Lie algebra $\mathfrak{n}$ by $\mathcal{C}^0\mathfrak{n} = \mathfrak{n}$ and $\mathcal{C}^{j+1}\mathfrak{n} = [\mathfrak{n}, \mathcal{C}^j\mathfrak{n}]$, $j \in \mathbb{N}$ we have the following

Definition 1. A Lie algebra $\mathfrak{n}$ is called *k-step nilpotent*, if $\mathcal{C}^k\mathfrak{n} = \{0\}$, but $\mathcal{C}^{k-1}\mathfrak{n} \neq \{0\}$ for some $k \in \mathbb{N}$.

An n-dimensional Lie algebra $\mathfrak{n}$ is called *filiform*, if it is $(n-1)$-step nilpotent. A filiform Lie algebra $\mathfrak{n}$ is *standard filiform*, if it contains a basis $\{G_1, \dots, G_n\}$ such that the nontrivial Lie bracket relations are given by $[G_1, G_i] = G_{i+1}$, $i = 2, \dots, n-1$.

Remark 1. For an n-dimensional filiform Lie algebra $\mathfrak{n}$ one has $\dim(\mathcal{C}^i\mathfrak{n}) = n - i - 1$ for $1 \le i \le n-1$. In any n-dimensional filiform Lie algebra $\mathfrak{n}$ there exists a basis $\{G_1, \dots, G_n\}$ such that $[G_1, G_i] = G_{i+1}$, $i = 2, \dots, n-1$, (cf. M. Vergne [8], D. M. Millionschikov [9], Lemma 3.4). A general filiform Lie algebra may have more non-trivial commutation relations, the simplest examples of filiform Lie algebras are the standard filiform Lie algebras.

On the other hand we deal with *nilpotent Lie algebras of dimension* ≤ 5 with nilpotency class > 2, which are not direct products of Lie algebras of lower dimension. According to [10], pp. 646–647, these Lie algebras are given up to isomorphism by the following non-vanishing commutators:

$$\begin{aligned}
&\mathfrak{l}_{4,3}: && [G_1, G_2] = G_3, && [G_1, G_3] = G_4; && && \\
&\mathfrak{l}_{5,5}: && [G_1, G_2] = G_4, && [G_1, G_4] = G_5, && [G_2, G_3] = G_5; && \\
&\mathfrak{l}_{5,6}: && [G_1, G_2] = G_3, && [G_1, G_3] = G_4, && [G_1, G_4] = G_5, && [G_2, G_3] = G_5; \\
&\mathfrak{l}_{5,7}: && [G_1, G_2] = G_3, && [G_1, G_3] = G_4, && [G_1, G_4] = G_5; && \\
&\mathfrak{l}_{5,9}: && [G_1, G_2] = G_3, && [G_1, G_3] = G_4, && [G_2, G_3] = G_5, &&
\end{aligned}$$

with respect to a distinguished basis $\{G_1, G_2 \dots\}$, which will be called the *canonical basis* of the corresponding Lie algebra. In this list of Lie algebras $\mathfrak{l}_{4,3}$ and $\mathfrak{l}_{5,7}$ are standard filiform, $\mathfrak{l}_{5,6}$ is non-standard filiform, $\mathfrak{l}_{5,5}$ and $\mathfrak{l}_{5,9}$ are 3-step nilpotent with 1-dimensional, respectively 2-dimensional center.

A Lie algebra equipped with an inner product is called *metric Lie algebra*, the automorphisms preserving the inner product are called *orthogonal automorphisms*.

In the following $\mathbb{E}^n$ denotes an n-dimensional Euclidean vector space with a distinguished orthonormal basis $\mathcal{E} = \{E_1, E_2, \dots, E_n\}$. We will use the following heuristic procedure for the classification of metric Lie algebras up to isometric isomorphisms:

1. Let $\{G_1, G_2, \dots, G_n\}$ be a fixed basis of an n-dimensional Lie algebra $\mathfrak{n}$ such that the commutation relations have a simple form (e.g. as in the list of the classification of low dimensional Lie algebras in the previous list).

2. Using the Gram–Schmidt process to the ordered basis $(G_n, G_{n-1}, \dots, G_1)$ in the metric Lie algebra $(\mathfrak{n}, \langle ., . \rangle)$ we obtain an orthonormal basis

$\{F_1, F_2, \dots, F_n\}$ expressed by

$$F_i = \sum_{k=i}^{n} a_{ik} G_k, \quad a_{ik} \in \mathbb{R}, \quad \text{such that} \quad a_{ii} \ge 0.$$

Conversely, any basis $\{F_1, F_2, \dots, F_n\}$ of $\mathfrak{n}$ having the form $F_i = \sum_{k=i}^{n} a_{ik} G_k$, $a_{ik} \in \mathbb{R}$ with $a_{ii} \ge 0$ determines an inner product on $\mathfrak{n}$ as an orthonormal basis. Such bases parametrize the inner products on $\mathfrak{n}$.

3. We define a Lie bracket on $\mathbb{E}^n$ with the same structure coefficients with respect to its distinguished basis $\mathcal{E}$ as the metric Lie algebra $(\mathfrak{n}, \langle ., . \rangle)$ has with respect to its basis $\mathcal{F}$. The obtained metric Lie algebra on $\mathbb{E}^n$ is depending on real parameters, (determined by the structure coefficients), and it is isometrically isomorphic to $(\mathfrak{n}, \langle ., . \rangle)$.