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The calculus of multivectors on noncommutative jet spaces

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ABSTRACT

The Leibniz rule for derivations is invariant under cyclic permutations of co-multiples within the arguments of derivations. We explore the implications of this principle: in effect, we construct a class of noncommutative bundles in which the sheaves of algebras of walks along a tessellated affine manifold form the base, whereas the fibres are free associative algebras or, at a later stage, such algebras quotients over the linear relation of equivalence under cyclic shifts. The calculus of variations is developed on the infinite jet spaces over such noncommutative bundles.

In the frames of such field-theoretic extension of the Kontsevich formal noncommutative symplectic (super)geometry, we prove the main properties of the Batalin–Vilkovisky Laplacian and Schouten bracket. We show as by-product that the structures which arise in the classical variational Poisson geometry of infinite-dimensional integrable systems do actually not refer to the graded commutativity assumption.

0. Introduction

Let $\mathbb{F}$ be a free algebra over $\Bbbk := \mathbb{R}$ and suppose $a_1, \ldots, a_k \in \mathbb{F}$. Denote by $\circ$ the associative multiplication in $\mathbb{F}$ and by $\mathfrak{t}$ the counterclockwise cyclic shift of co-multiples in the product $a_1 \circ \ldots \circ a_k$,

$$\mathfrak{t}(a_1 \circ \ldots \circ a_{k-1} \circ a_k) \stackrel{\text{def}}{=} a_k \circ a_1 \circ \ldots a_{k-1}.$$

For the sake of definition, now assume that a given derivation $\partial : \mathbb{F} \to \mathbb{F}$ is such that its values at $a_1, \ldots, a_k$ do not leave that set. By the Leibniz rule, the derivation is cyclic-shift invariant:

$$\partial\big(\mathfrak{t}(a_1 \circ \ldots \circ a_k)\big) = \mathfrak{t}\left(\partial(a_1 \circ \ldots \circ a_k)\right). \tag{1}$$

Indeed, both sides of the above equality are given by the sum

$$\partial(a_k) \circ a_1 \circ \ldots \circ a_{k-1} + a_k \circ \partial(a_1) \circ \ldots \circ a_k + a_k \circ a_1 \circ \ldots \circ \partial(a_{k-1}),$$

up to a sequential order in which these k summands follow each other (see Fig. 1). This observation is generalised in an obvious way to the case where the elements of algebra $\mathbb{F}$ are graded by some Abelian group, each element $a_1, \ldots, a_k$ is homogeneous with respect to the grading, and $\partial : \mathbb{F} \to \mathbb{F}$ is a graded derivation (i.e. not necessarily preserving the set $\{a_1, \ldots, a_k\}$ at hand).

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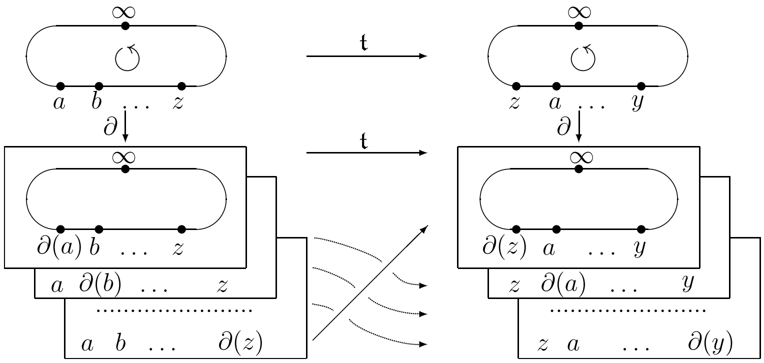

Fig. 1. The cyclic-shift invariance of derivations.

How much (graded-) commutativity is really needed to make the calculus of variations in the Lagrangian and Hamiltonian formalisms work, thus allowing for the Batalin–Vilkovisky technique for quantisation of gauge systems — and creating a cohomological approach to the complete integrability of infinite-dimensional KdV-type systems?[1]

We claim that it is not the restrictive assumption of commutativity that shows through *arbitrary* permutations — but it is the linear equivalence $a \sim \mathfrak{t}(a)$ of words a, written in a given alphabet, with respect to the *cyclic* permutations $\mathfrak{t}$ that is sufficient for the structures of the calculus of iterated variations to be well defined. Introduced in this cyclic-invariant setup, the Batalin–Vilkovisky Laplacian $\varDelta$ and variational Schouten bracket $[\![\,,\,]\!]$ are proven to satisfy the main identities such as the cocycle condition $\varDelta^2 = 0$, see (2a)–(2d). Both the definitions and assertions are then literally valid in the sub-class of graded-commutative geometries; the reason why is that the latter can be obtained from the former by using the postulated commutativity reduction at the end of the day when the proof is over.

The idea to establish the formal noncommutative symplectic geometry on the cyclic invariance, generalising the geometry of commutative symplectic manifolds, was introduced by Kontsevich in [20], cf. [21] and references therein. The quotient spaces of cyclic words were employed as target sets for maps from usual manifolds in [17] by Olver and Sokolov (cf. Model 1); several integrable equations of KdV-type were recovered in such noncommutative set-up.[2] Variations arise in the Poisson or Schouten brackets for integral functionals, their calculus was then pursued along the lines of [16]. The paper [17] initiated a classification and study of evolutionary ODE and PDE systems on associative algebras, which required the calculation of standard geometric structures for such models in jet spaces (e.g., see [22] in this context).

In this paper we further that approach to noncommutative jet spaces.[3] Continuing the line of reasoning from [9,26,27] where the intrinsic regularisation of Batalin–Vilkovisky formalism is revealed, we verify the main identities for $\varDelta$ and $[\![\,,\,]\!]$ in the variational noncommutative set-up of (homogeneous) local functionals F, G, H:

$$\varDelta(F \times G) = \varDelta F \times G + (-)^{|F|}[\![F, G]\!] + (-)^{|F|} F \times \varDelta G, \tag{2a}$$

$$[\![F, G \times H]\!] = [\![F, G]\!] \times H + (-)^{(|F|-1)\cdot|G|} G \times [\![F, H]\!], \tag{2b}$$

$$\varDelta\big([\![F, G]\!]\big) = [\![\varDelta F, G]\!] + (-)^{|F|-1}[\![F, \varDelta G]\!], \tag{2c}$$

$$\text{Jacobi}\big([\![\,,\,]\!]\big) = 0 \qquad \Longleftrightarrow \qquad \varDelta^2 = 0. \tag{2d}$$

It is quite paradoxical that for a long time, these identities were proclaimed to be valid just formally [5,28]; for it was believed that the Batalin–Vilkovisky technique would necessarily contain some divergencies or "infinite constants", whereas their manual regularisation appealed to surreal principles like "$\boldsymbol{\delta}(0) := 0$" for the Dirac $\boldsymbol{\delta}$-function (see [9] and references therein for discussion on the history of the problem).

The notion of associative algebra structures itself has deserved much attention in the mathematical physics literature, e.g., in relation to the Yang–Baxter equation. Such structures arise naturally in the topological context; the calculus of cyclic

1 We refer to [1–8] or [9] and to [10–17] respectively (see also [18,19] in both contexts).

2 Noncommutative extensions of classical infinite-dimensional systems can acquire new components that are invisible in the commutative world: e.g., there appear – often, through nonlocalities – the terms that contain the commutants $a_i \circ a_j - a_j \circ a_i$.

3 We note that the positive differential order calculus on infinite jet spaces lies far beyond the bare tensor calculus on usual commutative manifolds; for instance, compare [23] with [19] or contrast [24] vs [25] and [6] vs [9].