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Estimates for eigenvalues of the Wentzell–Laplace operator

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ABSTRACT

For an n-dimensional compact submanifold M with boundary in the Euclidean space $\mathbb{R}^N$, we give some sharp estimates for eigenvalues of the Wentzell–Laplace operator of M.

1. Introduction

Let (M, g) be an n-dimensional compact Riemannian manifold with boundary ∂M, and let $\bar{\Delta}$ and Δ be the Laplace–Beltrami operators on M and ∂M, respectively. Assume that τ is a real number and consider the eigenvalue problem for Wentzell boundary conditions

$$\begin{cases} \bar{\Delta} u = 0, & \text{in } M, \\ -\tau \Delta u + \dfrac{\partial u}{\partial \nu} = \lambda u, & \text{on } \partial M, \end{cases} \tag{1.1}$$

where ν denotes the outward unit normal vector field of ∂M. When M is a bounded domain in a Euclidean space, the above problem has been studied recently in [1]. A general derivation of Wentzell boundary conditions can be found in [2]. More recently, Xia and Wang [3] gave some estimates for eigenvalues of problem (1.1) when M is a Riemannian manifold.

When $\tau = 0$, the eigenvalue problem (1.1) becomes the following Steklov eigenvalue problem:

$$\bar{\Delta} u = 0 \ \text{ in } M, \quad \frac{\partial u}{\partial \nu} = pu \ \text{ on } \partial M. \tag{1.2}$$

It is known that the Steklov problem (1.2) has a discrete spectrum

$$0 = p_0 < p_1 \le p_2 \le \cdots \le p_k \le \cdots .$$

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The study of this problem was started by Steklov [4], whose motivation came from physics. The function u represents the steady state temperature on M such that the flux on the boundary is proportional to the temperature. From then on, for the Steklov eigenvalue problem, many interesting results have been obtained (cf. [5–15]).

When $\tau \geq 0$, the spectrum of the Laplacian with Wentzell condition consists in an increasing countable sequence of eigenvalues

$$0 = \lambda_{0,\tau} < \lambda_{1,\tau} \leq \lambda_{2,\tau} \leq \cdots \leq \lambda_{k,\tau} \leq \cdots$$

with corresponding real orthonormal (in $L^2(\partial M)$) eigenfunctions $u_0, u_1, u_2, \ldots$. Throughout the paper, we assume that $\tau \geq 0$.

We adopt the convention that each eigenvalue is repeated according to its multiplicity. Consider the Hilbert space

$$H(M) = \left\{u \in H^1(M), \mathrm{Tr}_{\partial M}(u) \in H^1(\partial M)\right\}, \tag{1.3}$$

where $\mathrm{Tr}_{\partial M}$ is the trace operator. We define on $H(M)$ two bilinear forms

$$B_\tau(u, v) = \int_M \langle \bar{\nabla} u, \bar{\nabla} v\rangle dV + \tau \int_{\partial M} \langle \nabla u, \nabla v\rangle dA, \quad D(u, v) = \int_{\partial M} uv dA, \tag{1.4}$$

where $\bar{\nabla}$ and ∇ are the gradient operators on M and ∂M, respectively. Since τ is nonnegative, the two bilinear forms are positive and the variational characterization for the kth eigenvalue is

$$\lambda_{k,\tau} = \min\left\{\frac{B_\tau(v, v)}{D(v, v)} : v \in H(M), v \neq 0, \int_{\partial M} vu_j dA = 0, j = 1, \ldots, k-1\right\}, k = 1, 2, \ldots. \tag{1.5}$$

In particular, when $k = 1$, the minimum is taken over the functions orthogonal to the eigenfunctions associated to $\lambda_{0,\tau} = 0$, i.e., the constant functions.

In this paper, we firstly give a Reilly-type inequality for eigenvalues of the problem (1.1) which generalizes the corresponding result of the Steklov eigenvalues in [16]. Before stating the result, let us fix the notion of the higher order mean curvature. One can also find it in [17]. Let (W, g) be an m-dimensional submanifold immersed in an N-dimensional space form $\mathbb{Q}^N(c)$, $c = 0, 1$, i.e., $\mathbb{Q}^N(0)$ is the Euclidean space $\mathbb{R}^{m+p}$ and $\mathbb{Q}^N(1)$ is the unit sphere $\mathbb{S}^N$. Denote by $y : W \to \mathbb{Q}^N(c)$ the immersion and let $\mathbf{h}$ be the second fundamental form of W, which is normal-vector valued. Suppose that $\{e_i\}_{i=1}^m$ is a local orthonormal basis for the tangent bundle of W with dual $\{\theta_i\}_{i=1}^m$, and moreover, $\{e_A\}_{A=m+1}^N$ is a local orthonormal basis for the normal bundle of W. Let $\mathbf{h}_{ij} := \mathbf{h}(e_i, e_j) = \sum_{A=m+1}^N h_{ij}^A e_A$, where h_{ij}^A, $i, j = 1, \ldots, m$, $A = m+1, \ldots, N$, are the components of the second fundamental form of W. Clearly, $(\mathbf{h}_{ij})$ is a vector matrix with respect to the frame $\{e_i\}_{i=1}^m$. One can define a $(0, 2)$-tensor T_r for $r \in \{0, 1, \ldots, m-1\}$ as follows. If r is even, we set

$$\begin{aligned} T_r &= \frac{1}{r!}\sum_{\substack{i_1\cdots i_r i\\ j_1\cdots j_r j}} \delta_{j_1\cdots j_r j}^{i_1\cdots i_r i}\left\langle \mathbf{h}_{i_1 j_1}, \mathbf{h}_{i_2 j_2}\right\rangle \cdots \left\langle \mathbf{h}_{i_{r-1} j_{r-1}}, \mathbf{h}_{i_r j_r}\right\rangle \theta_i \otimes \theta_j \\ &= \sum_{i,j} T_{rj}^i \theta_i \otimes \theta_j, \end{aligned}$$

where $\delta_{j_1\cdots j_r j}^{i_1\cdots i_r i}$ $(1 \leq i_1, \ldots, i_r, i \leq m, 1 \leq j_1, \ldots, j_r, j \leq m)$ are the generalized Kronecker symbols and

$$T_{rj}^i = \frac{1}{r!}\sum_{\substack{i_1\cdots i_r i\\ j_1\cdots j_r j}} \delta_{j_1\cdots j_r j}^{i_1\cdots i_r i}\left\langle \mathbf{h}_{i_1 j_1}, \mathbf{h}_{i_2 j_2}\right\rangle \cdots \left\langle \mathbf{h}_{i_{r-1} j_{r-1}}, \mathbf{h}_{i_r j_r}\right\rangle.$$

We also set

$$\begin{aligned} T_{r-1} &= \frac{1}{(r-1)!}\delta_{j_1\cdots j_r j}^{i_1\cdots i_r i}\left\langle \mathbf{h}_{i_1 j_1}, \mathbf{h}_{i_2 j_2}\right\rangle \cdots \left\langle \mathbf{h}_{i_{r-3} j_{r-3}}, \mathbf{h}_{i_{r-2} j_{r-2}}\right\rangle \mathbf{h}_{i_{r-1} j_{r-1}} \theta_i \otimes \theta_j \\ &= \left(T_{(r-1)ij}^A e_A\right) \theta_i \otimes \theta_j, \end{aligned}$$

where

$$T_{(r-1)ij}^A = \frac{1}{(r-1)!}\delta_{j_1\cdots j_r j}^{i_1\cdots i_r j}\left\langle \mathbf{h}_{i_1 j_1}, \mathbf{h}_{i_2 j_2}\right\rangle \cdots \left\langle \mathbf{h}_{i_{r-3} j_{r-3}}, \mathbf{h}_{i_{r-2} j_{r-2}}\right\rangle h_{i_{r-1} j_{r-1}}^A.$$

For any even integer $r \in \{0, 1, \ldots, m-1\}$, the rth mean curvature function $\mathbf{H}_r$ and $(r+1)$-th mean curvature vector field $\mathbf{H}_{r+1}$ are defined by

$$\mathbf{H}_r = \frac{1}{c_n^r r} T_{(r-1)ij}^A h_{i_{r-1} j_{r-1}}^A, \tag{1.6}$$

$$\mathbf{H}_{r+1} = \frac{1}{c_n^{r+1}(r+1)} T_{rj}^i h_{i_{r-1} j_{r-1}}^A e_A, \tag{1.7}$$