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Fuzzy circle and new fuzzy sphere through confining potentials and energy cutoffs

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ABSTRACT

Guided by ordinary quantum mechanics we introduce new fuzzy spheres S^d_Λ of dimensions $d = 1, 2$: we consider an ordinary quantum particle in $D = d + 1$ dimensions subject to a rotation invariant potential well $V(r)$ with a very sharp minimum on a sphere of unit radius. Imposing a sufficiently low energy cutoff to 'freeze' the radial excitations makes only a finite-dimensional Hilbert subspace accessible and on it the coordinates noncommutative *à la Snyder*; in fact, on it they generate the whole algebra of observables. The construction is equivariant not only under rotations – as Madore's fuzzy sphere – , but under the full orthogonal group $O(D)$. Making the cutoff and the depth of the well dependent on (and diverging with) a natural number Λ, and keeping the leading terms in $1/\Lambda$, we obtain a sequence S^d_Λ of fuzzy spheres converging to the sphere S^d in the limit $\Lambda \to \infty$ (whereby we recover ordinary quantum mechanics on S^d). These models may be useful in condensed matter problems where particles are confined on a sphere by an (at least approximately) rotation-invariant potential, beside being suggestive of analogous mechanisms in quantum field theory or quantum geometry.

1. Introduction

In 1947 Snyder proposed [1] the first example of noncommutative spacetime hoping that nontrivial (but Poincaré equivariant) commutation relations among the coordinates, acting as a fundamental regularization procedure, could cure ultraviolet (UV) divergencies in quantum field theory (QFT).[1] He dubbed as *distasteful arbitrary* the UV regularization based on momentum (or equivalently energy) cutoff, which had just been proposed in the literature at the time, presumably as it broke manifest Lorentz equivariance and looked *ad hoc*. Ironically, shortly afterwards this and other more sophisticated regularization procedures found widespread application within the renormalization method; as known, the latter has proved to be extremely successful in extracting physically correct predictions from quantum electrodynamics, chromodynamics, and more generally the Standard Model of elementary particle physics. The proposal of Snyder was thus almost forgotten for decades (exceptions are e.g. [3,4]). On the other hand, it is believed that any consistent quantum theory of gravitation will set fundamental bounds of the order of Planck length $l_p = \sqrt{\hbar G/c^3} \sim 10^{-33}$cm on the accuracy Δx of localization measurements. The arguments, which in qualitative form go back at least to [5–7], are based on a cutoff on the concentration

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[1] The idea had originated in the '30s from Heisenberg, who proposed it in a letter to Peierls [2]; the idea propagated via Pauli to Oppenheimer, who asked his student Snyder to develop it.

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of energy[2] ; they were made more precise and quantitative by Doplicher, Fredenhagen, Roberts [8], who also proposed that such a bound could follow from appropriate noncommuting coordinates (for a review of more recent developments see e.g. [9]). More generally, Connes' Noncommutative Geometry framework [10] allows not only to replace the commutative algebra $\mathcal{A}$ of functions on a manifold M by a noncommutative one, but also to develop on it the whole machinery of differential geometry [10,11]. Often one deals with a family of noncommutative deformations $\mathcal{A}_\lambda$ of $\mathcal{A}$ that become commutative in some limit of the family's ruling parameter(s) λ, exactly as the algebra of observables in ordinary quantum mechanics becomes the algebra of functions on phase space as $\hbar \to 0$.

Fuzzy spaces are particular examples parametrized by a positive integer n, so that the algebra $\mathcal{A}_n$ is a finite-dimensional matrix algebra with dimension which increases and diverges with n while $\mathcal{A}_n \to \mathcal{A}$ (in a suitable sense). Since their introduction they have raised big interest in the high energy physics community as a non-perturbative technique in QFT (or string theory) based on a finite-discretization of space(time) alternative to the lattice one: the advantage is that the algebras $\mathcal{A}_n$ can carry representations of Lie groups (not only of discrete ones). The first and seminal fuzzy spaces are the Fuzzy Sphere (FS) of Madore and Hoppe [12,13] and the noncommutative torus [14,15] parametrized by a root of unity (this is often called fuzzy torus by theoretical physicists, see e.g. [16])[3] ; the first applications to QFT models of the FS are in [17,18]. The FS is a sequence of $SO(3)$-equivariant, finite noncommutative $*$-algebras $\mathcal{A}_n$ isomorphic to M_n (the algebra of $n \times n$ matrices); each matrix represents the expansion in spherical harmonics of an element of $C(S^2)$ truncated at level n. $\mathcal{A}_n$ is generated by hermitean noncommutative coordinates x^i fulfilling

$$[x^i, x^j] = \frac{2i}{\sqrt{n^2-1}}\varepsilon^{ijk}x^k, \qquad r^2 := x^i x^i = 1, \qquad n \in \mathbb{N}\setminus\{1\} \tag{1}$$

(here and below sum over repeated indices is understood). The Hilbert space is chosen as $\mathcal{H} \simeq \mathbb{C}^n$ so that it carry an irreducible representation of $U so(3)$, and the square distance r^2 from the origin – which is central – be identically equal to 1. We note that however (1) are equivariant only under $SO(3)$, not $O(3)$; in particular not under parity $x^i \mapsto -x^i$. Fuzzy spaces can be used also in extra dimensions to account for internal (e.g. gauge) degrees of freedom, see e.g. [19].

As the arguments leading to $\Delta x \geq l_p$ suggest, imposing an energy cutoff $\overline{E}$ on an existing theory can be physically justified by two reasons, at least. It may be a necessity when we believe that $\overline{E}$ represents the threshold for the onset of new physics not accountable by that theory. Or it may serve to yield an effective description of the system when we, as well as the interactions with the environment, are not able to bring its state to higher energies; this leads also to a lower bound in the accuracy with which our apparatus can measure some observables (position, momentum, …) of the system, which corresponds to the maximum energy transferable to the system by the apparatus in the measurement process, or by the environment during the interaction time. (Of course, the two reasons may co-exist.) Mathematically, the cutoff is imposed by a projection on the Hilbert subspace characterized by energies $E \leq \overline{E}$. If the Hamiltonian is invariant under some symmetry group, then the projection is invariant as well, and the projected theory will inherit that symmetry.

That imposing such a cutoff can modify a quantum mechanical model by converting its commuting coordinates into non-commuting ones is simply illustrated by the well-known Landau model (see e.g. [20–23]), which describes a charged quantum particle in 2D interacting only with an uniform magnetic field (in the orthogonal direction) B. The energy levels are $E_n = \frac{\hbar|eB|}{mc}n$, if we fix the additive constant so that the lowest level is $E_0 = 0$; choosing $\overline{E} \leq \frac{\hbar|eB|}{mc}$ (this may be physically justified e.g. by a very strong B), then the Hilbert space of states is projected to the subspace $\mathcal{H}_0$ of zero energy, and $\frac{e}{c}Bx, y$ become canonically conjugates, i.e. have a non-zero (but constant) commutator. The dimension of $\mathcal{H}_0$ is approximately proportional to the area of the surface, hence is finite (resp. infinite) if the area is.

Inspired by the projection mechanism in the Landau model, here we consider a quantum particle in dimension $D = 2$ or $D = 3$ with a Hamiltonian consisting of the standard kinetic term and a rotation invariant potential energy $V(r)$ with a very deep minimum (a well) respectively on a circle or on a sphere of unit radius; $k \equiv V''(1)/4 > 0$ plays the role of confining parameter. Imposing an energy cutoff $\overline{E}$ makes only a finite-dimensional Hilbert subspace $\mathcal{H}_{\overline{E}}$ accessible and the projected coordinates noncommutative on $\mathcal{H}_{\overline{E}}$. We choose $\overline{E} < 2\sqrt{2k}$ so that $\mathcal{H}_{\overline{E}}$ does not contain excited radial modes, and on it the Hamiltonian reduces to the square angular momentum; this can be considered as a quantum version of the constraint $r = 1$. It turns out that the coordinates generate the whole algebra $\mathcal{A}_{\overline{E}} := End\left(\mathcal{H}_{\overline{E}}\right)$ of observables on $\mathcal{H}_{\overline{E}}$. Their commutators are of Snyder type, i.e. proportional to the angular momentum components L_{ij} (apart from a small correction – depending only on the square angular momentum – on the highest energy states), rather than some function of the coordinates. Moreover, $(\mathcal{A}_{\overline{E}}, \mathcal{H}_{\overline{E}})$ is equivariant under the *full* group $O(D)$ of orthogonal transformations, because both the starting quantum mechanical model on $\mathcal{L}^2(\mathbb{R}^D)$ and the cut-off procedure are. Actually, we prove the realization $\mathcal{A}_{\overline{E}} = \pi_{\overline{E}}\left[U so(D+1)\right]$, with $\pi_{\overline{E}}$ a suitable irreducible unitary representation of $U so(D+1)$ on $\mathcal{H}_{\overline{E}}$; as a consequence, $\mathcal{H}_{\overline{E}}$ carries a *reducible* representation of the subalgebra $U so(D)$ generated by the angular momentum components L_{ij}, more precisely

[2] In fact, by Heisenberg uncertainty relations to reduce the uncertainty Δx of the coordinate x of an event one must increase the uncertainty Δp_x of the conjugated momentum component by use of high energy probes; but by general relativity the associated concentration of energy in a small region would produce a trapping surface (event horizon of a black hole) if it were too large; hence the size of this region, and Δx itself, cannot be lower than the associated Schwarzschild radius, l_p.

[3] In [13] the FS was used in connection with the quantization of the membrane; its noncommutative differential geometry was first constructed by Madore in [12]. The algebra of the noncommutative torus (NT) is generated by unitary elements U, V fulfilling $UV = VUq$, with q on the unit circle; when q is a root of unity the irreducible representations are finite-dimensional, and the NT can be dubbed as *fuzzy*.