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Morse–Darboux lemma for surfaces with boundary

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ABSTRACT

We formulate and prove an analog of the classical Morse–Darboux lemma for the case of a surface with boundary.

1. Introduction

Throughout this paper the word *smooth* means C^∞ smooth. The aim of this paper is to prove the following theorem.

Theorem 1. *Let M be a 2D surface with an area form ω, and let $f : M \to \mathbb{R}$ be a smooth function. Let also $O \in \partial M$ be a regular point for f and a non-degenerate critical point for $f|_{\partial M}$. Then there exists a chart (p, q) centered at O such that $\omega = dp \wedge dq$, and $f = \alpha \circ S$, where $S = q + p^2$ or $S = q - p^2$. The function α of one variable is smooth in the neighborhood of the origin $0 \in \mathbb{R}$ and $\alpha'(0) \neq 0$. Furthermore, we have $q \geq 0$ wherever q is defined and the boundary ∂M satisfies the equation $q = 0$. (See Fig. 1.)*

Theorem 1 is closely related to the classical Morse–Darboux lemma. Let us recall the statement of that lemma.

Theorem 2. *Let M be a 2D surface with an area form ω, and let $f : M \to \mathbb{R}$ be a smooth function. Let also $O \in M \setminus \partial M$ be a non-degenerate critical point for f. Then there exists a chart (p, q) centered at O such that $\omega = dp \wedge dq$, and $f = \alpha \circ S$, where $S = pq$ or $S = p^2 + q^2$. The function α of one variable is smooth in the neighborhood of the origin $0 \in \mathbb{R}$ and $\alpha'(0) \neq 0$.*

The Morse–Darboux lemma is a particular case of Le lemme de Morse isochore, see [1], and also a particular case of Eliasson's theorem on the normal form for an integrable Hamiltonian system near a non-degenerate critical point, see [2,3]. The Morse–Darboux lemma is an important tool in topological hydrodynamics, see [4], and theory of integrable systems, see [5].

We expect that the result of the present paper will also be useful in 2D fluid dynamics. In particular, it gives a partial answer to Problem 5.6 from [6] on the asymptotical properties of measures on Reeb graphs.

This paper is organized as follows. In Section 2 we formulate Theorem 1′ which is equivalent to Theorem 1. The proof of Theorem 1′ is given in Section 4. Section 3 contains several lemmas useful for the proof of Theorem 1′.

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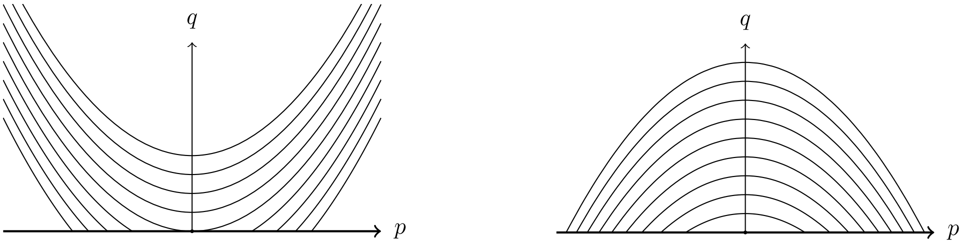

(a) Case $S = q + p^2$. (b) Case $S = q - p^2$.

Fig. 1. Level sets of f. The horizontal axis is the boundary of M.

2. Reformulation of the main theorem

Theorem 1′. *Let $\omega = \omega(x, y)dx \wedge dy$ be an area form on $\mathbb{R}^2$, and $f = f(x, y)$ be a smooth function such that $f_x(0, 0) = 0$, $f_y(0, 0) > 0$ and $f_{xx}(0, 0) > 0$. Then there exists a chart (p, q) centered at $(0, 0)$ such that $\omega = dp \wedge dq$, $f(p, q) = \alpha(p^2 + q)$, and $q = 0$ if and only if $y = 0$. The function α of one variable is smooth in the neighborhood of the origin $0 \in \mathbb{R}$ and $\alpha'(0) > 0$.*

Proposition 1. *Theorem 1 follows from Theorem 1′.*

Proof. Let us choose a chart (x, y) centered at O in ∂M such that $P \in \partial M$ if and only if $y(P) = 0$. The function $f(x, y)$ and the form $\omega(x, y)dx \wedge dy$ can be smoothly extended on some neighborhood of $(0, 0)$. As $(0, 0)$ is non-degenerate critical point for $f|_{\partial M}$ we have $f_x(0, 0) = 0, f_y(0, 0) \neq 0, f_{xx}(0, 0) \neq 0$. To fulfill conditions $f_y(0, 0) > 0, f_{xx}(0, 0) > 0$, we may need some of the following transformations: $f \to -f$, $y \to -y$. Now, we obtain the chart (p, q) from Theorem 1′. If $q \leq 0$ we need one more transformation: $q \to -q$, $p \to -p$. It remains to restrict the chart (p, q) to the upper half plane. □

3. Necessary lemmas

In this section we assume that conditions of Theorem 1′ hold. Also from now on we will assume that $f(0, 0) = 0$. This will simplify notation.

First of all, we want to prove an analog of the classical Morse Lemma for a surface with boundary.

Lemma 1. *There exists a chart $(\hat{x}, \hat{y})$ centered at $(0, 0)$ such that*

1. $\hat{x}(0, 0) = \hat{y}(0, 0) = 0$;
2. $f(\hat{x}, \hat{y}) = \hat{x}^2 + \hat{y}$;
3. $\hat{y}(x, y) = 0$ *if and only if* $y = 0$.

Proof. Hadamard's lemma implies that

$$f(x, y) = f_1(x, y)x + f_2(x, y)y,$$

where f_1 and f_2 are smooth functions, and $f_1(0, 0) = f_x(0, 0)$, $f_2(0, 0) = f_y(0, 0)$. Since $f_x(0, 0) = 0$ Hadamard's lemma similarly implies that

$$\begin{aligned} f(x, y) &= (f_{11}(x, y)x + f_{12}(x, y)y)x + f_2(x, y)y \\ &= f_{11}(x, y)x^2 + f_{12}(x, y)xy + f_2(x, y)y \\ &= (x\sqrt{f_{11}(x, y)})^2 + y(f_{12}(x, y)x + f_2(x, y)). \end{aligned}$$

Recall that $f_{xx}(0, 0) > 0$ and also notice that $f_{11}(0, 0) = \frac{1}{2}f_{xx}(0, 0)$. Consider the following transformation of coordinates

$$\hat{x}(x, y) = \sqrt{f_{11}(x, y)}x$$
$$\hat{y}(x, y) = y(f_{12}(x, y)x + f_2(x, y)).$$

The Jacobian determinant of this transformation at the point $(0, 0)$ is equal to $\sqrt{f_{11}(0, 0)}f_2(0, 0) > 0$. It follows from the inverse function theorem that functions $\hat{x}$ and $\hat{y}$ form a chart centered at $(0, 0)$. By construction

$$f(\hat{x}, \hat{y}) = \hat{x}^2 + \hat{y},$$

and $\hat{y}(x, y) = y(f_{12}(x, y)x + f_2(x, y)) = 0$ if and only if $y = 0$. □