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QUANTUM KOSZUL FORMULA ON QUANTUM SPACETIME

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ABSTRACT. Noncommutative or quantum Riemannian geometry has been proposed as an effective theory for aspects of quantum gravity. Here the metric is an invertible bimodule map $\Omega^1 \otimes_A \Omega^1 \to A$ where A is a possibly noncommutative or 'quantum' spacetime coordinate algebra and (Ω^1, d) is a specified bimodule of 1-forms or 'differential calculus' over it. In this paper we explore the proposal of a 'quantum Koszul formula' in [14] with initial data a degree -2 bilinear map $\perp$ on the full exterior algebra Ω obeying the 4-term relations

$$(-1)^{|\eta|}(\omega\eta) \perp \zeta + (\omega \perp \eta)\zeta = \omega \perp (\eta\zeta) + (-1)^{|\omega|+|\eta|}\omega(\eta \perp \zeta), \quad \forall \omega, \eta, \zeta \in \Omega$$

and a compatible degree -1 'codifferential' map δ. These provide a quantum metric and interior product and a canonical bimodule connection ∇ on all degrees. The theory is also more general than classically in that we do not assume symmetry of the metric nor that δ is obtained from the metric. We solve and interpret the $(\delta, \perp)$ data on the bicrossproduct model quantum spacetime $[r, t] = \lambda r$ for its two standard choices of Ω. For the α-family calculus the construction includes the quantum Levi-Civita connection for a general quantum symmetric metric, while for the more standard $\beta = 1$ calculus we find the quantum Levi-Civita connection for a quantum 'metric' that in the classical limit is *antisymmetric*. This suggests to consider quantum Riemannian and symplectic geometry on a more equal footing than is currently the case.

1. INTRODUCTION

Noncommutative differential geometry (NCDG) has been proposed for some three decades now as a natural generalisation of classical differential geometry that does not assume that the coordinate algebra or their differentials commute. There are many motivations and applications, many of them still unexplored (eg to actual quantum systems) but one of them is now widely accepted as an important role, namely as an effective theory for quantum gravity effects expressed as quantising spacetime itself. Of historical interest here was [20] in the 1940's, although this did not propose a closed spacetime algebra exactly but an embedding of it into something larger. Specific proposals relating to quantum gravity (the 'Planck scale Hopf algebra') appeared in [10] where they led to one of the two main classes of quantum groups to emerge in the 1980s as well as to one of the first and most well-studied quantum spacetimes with quantum group symmetry, namely the Majid-Ruegg 'bicrossproduct model' [15]. In 2D this is the coordinate algebra $[r, t] = \lambda r$ where λ should be $\imath$ times the Planck scale of around 10^{-35}m. In spite of many hundreds of papers on this quantum spacetime, it continues to be useful as a testbed

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