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Reduction in soliton hierarchies and special points of classical r-matrices

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ABSTRACT

We propose the most general approach to construction of the U–V pairs of hierarchies of soliton equations in two dimensions based on the theory of classical non-skew-symmetric r-matrices with spectral parameters and infinite-dimensional Lie algebras. We show that *reduction* in integrable hierarchies is connected with "special points" of classical r-matrices in which they become singular or degenerated. We prove that "Mikhailov's reduction" or reduction with the help of automorphism is a partial case of our construction. We consider two types of integrable hierarchies and the corresponding soliton equations: the so-called "positive" and "negative flow" equations. We show that the "negative flow" equations can be written in the most general case without a specification of the concrete form of classical r-matrix. They coincide with a generalization of chiral field equation and its different reductions, with a generalization of abelian and non-abelian Toda field equations and new class of integrable equations which we call "double-shift" equations. For the case of "positive" hierarchies and "positive" flows we explicitly write general U–V pairs of "nominative" equations of hierarchy. We consider examples of new equations of such the types and present new soliton equations coinciding with elliptic deformation of dNS equation and its "negative flow" equation and ultimate generalization of the abelian and non-abelian modified Toda field equations.

1. Introduction

The theory of soliton equations in two dimensions has been extensively developed during the last fifty years since pioneering work [1]. There have been a lot of advances in construction of new soliton equations and in understanding of mechanisms of their integrability (see e.g. classical reviews and books [2–4] or recent book [5] and references therein). Nevertheless till now there is no complete description and classification of the hierarchies of (many-component) soliton equations in two dimensions.

In the present paper we propose a way to solve this problem, developing the most general scheme that should lead to a description and classification of the hierarchies of soliton equations in the dimension two. It is known that integrability of such the PDE-s is based on the possibility to represent them in zero-curvature form [6]:

$$\frac{\partial U(x,t,u)}{\partial t} - \frac{\partial V(x,t,u)}{\partial x} + [U(x,t,u), V(x,t,u)] = 0, \tag{1}$$

where U, V are $\mathfrak{g}$-valued functions ($\mathfrak{g}$ is a simple or reductive Lie algebra), depending on the dynamical variables (fields), their derivatives with respect to the "space" coordinate x and an additional complex parameter u usually called "spectral".

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One of the most effective approaches to zero-curvature equation is to interpret them as a compatibility condition of two Euler–Arnold (generalized Lax) equations on dual space to some infinite-dimensional Lie algebra $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$-valued functions of spectral parameter u:

$$\frac{\partial L(u)}{\partial x} = ad^*_{U(x,t,u)}L(u), \quad \frac{\partial L(u)}{\partial t} = ad^*_{V(x,t,u)}L(u), \tag{2}$$

where $L(u)$ is a generic element of $\widetilde{\mathfrak{g}}^*$, which (as a space of $\mathfrak{g}$-valued functions of u) does not, generally speaking, coincide with $\widetilde{\mathfrak{g}}$. If $U(x, t, u)$ and $V(x, t, u)$ are the $\widetilde{\mathfrak{g}}$-valued gradients of some Poisson-commutative functions on $\widetilde{\mathfrak{g}}^*$ then the consistency of Eqs. (2) is automatically guaranteed and the corresponding U–V pair satisfies (1). That is, having infinite-dimensional Lie algebra of $\mathfrak{g}$-valued functions of spectral parameter possessing infinite set of Poisson-commuting functions on its dual space one automatically obtains infinite set of $\widetilde{\mathfrak{g}}$-valued U–V pairs satisfying zero-curvature equations, i.e. one automatically obtains integrable hierarchy. The Lie algebra $\widetilde{\mathfrak{g}}$ may be called in this case "the Lie algebra of U–V-pairs". Hence, in the framework of this approach the task of generation and classification of integrable PDE-s in two dimensions is reduced to the task of construction and classification of special infinite-dimensional Lie algebras of $\mathfrak{g}$-valued functions of u possessing infinite set of Poisson-commuting functions on their dual spaces.

The described above approach to soliton equations was independently proposed by three groups of authors [7–9] for the cases when $\widetilde{\mathfrak{g}}$ are subalgebras of loop algebra: $\widetilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathrm{Pol}(u)$ or $\widetilde{\mathfrak{g}} = \mathfrak{g} \otimes u^{-1}\mathrm{Pol}(u^{-1})$ which are known to possess the needed property [10,11]. The main question in this Lie-algebraic scheme is the following: how to construct the other, more complicated infinite-dimensional Lie algebras of $\mathfrak{g}$-valued functions of u possessing infinitely many Poisson-commuting functions on their dual spaces? In the present paper we give an answer to this question. For this purpose we unify the ideas of [7–9] with ideas of [12,13] and construct the needed infinite-dimensional Lie algebras starting with classical (non-skew-symmetric in general) r-matrices $r(u, v)$ with spectral parameters. The classical r-matrix appears in this theory quite naturally. Indeed, the Poisson commutativity of the spectral invariants of Lax equation is necessarily connected with the existence of the classical r-matrix [14]. As far as we are dealing with a set of Lax-type equations and require the commutativity of the corresponding hamiltonians that generate the Lax-type flows, the appearance of the r-matrix is inevitable. That is why it is quite natural to start with the r-matrix, encoding it in the structure of the algebra $\widetilde{\mathfrak{g}}$.

Let us note that for the needs of the theory of soliton equations the so-called "non-dynamical" classical r-matrix satisfying the following "generalized" classical Yang–Baxter equation [14–16]:

$$[r_{12}(u_1, u_2), r_{13}(u_1, u_3)] = [r_{23}(u_2, u_3), r_{12}(u_1, u_2)] - [r_{32}(u_3, u_2), r_{13}(u_1, u_3)]$$

is pertinent. In the case of skew-symmetric r-matrices this equation passes to the usual classical Yang–Baxter equation [3,17], but neither from the point of view of classical integrable systems [18] nor from the point of view of the quantum ones [19] the restriction to skew-symmetric case is compulsory.

The procedure of "extracting" of special infinite-dimensional Lie algebra from the classical r-matrix is non-trivial. In our previous papers [20,21] we have performed this procedure for the case of the "regular" points ν of the classical r-matrices in the vicinity of which the r-matrix is non-degenerated, generalizing in such a way the results of [12,13] from the case of skew-symmetric r-matrices onto the general non-skew-symmetric case. The obtained in such a way infinite-dimensional Lie algebras are denoted by $\widetilde{\mathfrak{g}}_r^{-,\nu}$. They are Lie algebras of special $\mathfrak{g}$-valued meromorphic functions with the poles in the point ν. The corresponding Lax matrix $L(u)$ (that will be denoted by $L^{-,\nu}(u)$) as an element of the dual space $(\widetilde{\mathfrak{g}}_r^{-,\nu})^*$ is identified with a generic element of the space of $\mathfrak{g}$-valued formal Taylor series $\mathfrak{g}((u-\nu))$ and satisfies the linear r-matrix bracket:

$$\{L_1^{-,\nu}(u), L_2^{-,\nu}(v)\} = [r_{12}(u, v), L_1^{-,\nu}(u)] - [r_{21}(v, u), L_1^{-,\nu}(u)].$$

The corresponding commuting integrals (coefficients by different degrees of u in decompositions of the traces of powers of Lax matrix $L^{-,\nu}(u)$) are finite polynomials in the coordinate functions on $(\widetilde{\mathfrak{g}}_r^{-,\nu})^*$. Their $\widetilde{\mathfrak{g}}_r^{-,\nu}$-valued gradients produce the needed U–V pairs for the integrable hierarchy associated with a given classical r-matrix. The simplest of the constructed U–V pairs is U–V pair of the generalized Heisenberg magnet/Landau–Lifshitz equations defined in a generic fashion for arbitrary classical r-matrix [21]. Considering in a similar way "two-poled" r-matrix algebra $\widetilde{\mathfrak{g}}_r^{-,\nu_1} + \widetilde{\mathfrak{g}}_r^{-,\nu_2}$ with both poles ν_1, ν_2 being regular points of the classical r-matrix, we have obtained "doubled" integrable hierarchy containing negative flows of the generalized Heisenberg magnet/Landau–Lifshitz hierarchies [21]. The simplest "nominative" equation of this hierarchy is a generalization of chiral field hierarchy [22] onto the cases of arbitrary non-skew-symmetric classical r-matrices [21].

There arises the following natural question: where in this general r-matrix picture all other soliton hierarchies lie? In the present paper we answer this question. For this purpose we note that all other known integrable hierarchies are one-component or vector-type integrable hierarchies, while generalized Heisenberg magnet/Landau–Lifshitz hierarchies and generalized chiral field hierarchy are matrix hierarchies. That is, in order to obtain one-component or vector-type integrable hierarchies one has to perform a kind of a reduction procedure in the matrix hierarchies and in the corresponding algebras of U–V pairs. In the present paper we propose a simple and effective way how to do this using the very classical r-matrix $r(u, v)$. It occurred that in order to perform the mentioned *reduction* in integrable hierarchies it is necessary to consider "special" or "singular" points $v = \nu_0$ in vicinity of which the r-matrix becomes degenerated or singular. This leads to the change in structure of the r-matrix algebra $\widetilde{\mathfrak{g}}_r^{-,\nu_0}$ which stops to be isomorphic to the Lie algebra $\widetilde{\mathfrak{g}}_r^{-,\nu}$ corresponding to regular points ν. The structure of the dual space $(\widetilde{\mathfrak{g}}_r^{-,\nu_0})^*$ also changes. As a consequence of this the U–V pairs of the corresponding hierarchy include less independent coordinate functions on $(\widetilde{\mathfrak{g}}_r^{-,\nu_0})^*$ for which the resulting soliton equations are written,