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Contravariant form for reduction algebras

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ABSTRACT

We define contravariant forms on diagonal reduction algebras, algebras of $\mathbf{h}$-deformed differential operators and on standard modules over these algebras. We study properties of these forms and their specializations. We show that the specializations of the forms on the spaces of $\mathbf{h}$-commuting variables present zero singular vectors iff they are in the kernel of the specialized form. As an application we compute norms of highest weight vectors in the tensor product of an irreducible finite dimensional representation of the Lie algebra $\mathbf{gl}_n$ with a symmetric or wedge tensor power of its fundamental representation.

1. Introduction

The contravariant (or Shapovalov) form on highest weight modules is a powerful tool in representation theory of reductive Lie algebra. It is used for the construction of irreducible representations, description of singular vectors of Verma modules etc. [1,2]. In this paper we define and compute an analogue of the Shapovalov form for certain modules over reduction algebras. As an illustration we calculate the norms of singular vectors in tensor product of irreducible finite-dimensional representation of Lie algebra $\mathbf{gl}_n$ and symmetric or exterior powers of its fundamental representation.

Let $\mathbf{g}$ be a reductive Lie algebra with a given triangular decomposition $\mathbf{g} = \mathbf{n}_- + \mathbf{h} + \mathbf{n}_+$. Here $\mathbf{n}_\pm$ are two opposite maximal nilpotent subalgebras of $\mathbf{g}$ and $\mathbf{h}$ is the Cartan subalgebra. The reduction algebras $\bar{\mathrm{Z}}_\pm$ and $\bar{\mathrm{Z}}$ (the latter is called sometimes the double coset algebra) are built from the pair $(\mathcal{A}, \mathrm{U}(\mathbf{g}))$, where $\mathcal{A}$ is an associative algebra which contains the universal enveloping algebra $\mathrm{U}(\mathbf{g})$ of $\mathbf{g}$, see Section 2.1 for details. We define contravariant form for three particular double coset reduction algebras: the diagonal reduction algebra $\bar{\mathcal{D}}(\mathbf{gl}_n)$, and the algebra $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ together with its odd version $\mathcal{G}\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$, see Section 2.3 for definitions.

The algebraic construction of the contravariant form for universal enveloping algebras of semisimple Lie algebras is based on the Harish-Chandra map [3] whose construction uses the triangular decomposition $\mathbf{g} = \mathbf{n}_- + \mathbf{h} + \mathbf{n}_+$ which implies that $\mathrm{U}(\mathbf{g})$ decomposes into a sum of a left ideal $\mathbf{n}_-\,\mathrm{U}(\mathbf{g})$, right ideal $\mathrm{U}(\mathbf{g})\mathbf{n}_+$ and the commutative ring $\mathrm{U}(\mathbf{h})$. The Harish-Chandra map and the contravariant form can be defined for the algebras $\bar{\mathcal{D}}(\mathbf{gl}_n)$, $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ and $\mathcal{G}\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ as well. Note that there is no analogue of the triangular decomposition for the diagonal reduction algebra $\bar{\mathcal{D}}(\mathbf{gl}_n)$. However, the diagonal reduction algebra possesses a similar to above decomposition into the sum of a certain left ideal, right ideal and the commutative ring

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over the localized universal enveloping algebra $\bar{\mathrm{U}}(\mathbf{h})$. We use this decomposition for the definition of an analogue of the Harish-Chandra map and then for the definition of the contravariant form.

The next step is to extend the constructed form to certain "standard" modules over reduction algebras, see Section 3.3 for details. A standard module possesses a $(\bar{\mathrm{Z}}, \bar{\mathrm{U}}(\mathbf{h}))$-bimodule structure and, besides, can be regarded as a deformation of the space of rational functions on $\mathbf{h}^*$ with values in V where V is a $\mathrm{U}(\mathbf{g})$-module in the case of diagonal reduction algebra or the module over the ring of differential operators in the case of reduction algebras $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ and $\mathcal{G}\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$. The specialization of a standard module over the diagonal reduction algebra to a generic weight $\lambda \in \mathbf{h}^*$ can be identified with the space of intertwining operators, see e.g. [4],

$$M_\lambda \to V \otimes M_\mu,$$

equipped with a structure of a module over the diagonal reduction algebra. Here M_λ and M_μ are Verma modules over $\mathbf{g}$. The diagonal reduction algebra $\bar{\mathcal{D}}(\mathbf{gl}_n)$ and the algebras $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ and $\mathcal{G}\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ can be regarded as deformations, in the above sense, of the algebras $\mathrm{U}(\mathbf{gl}_n)$ and, respectively, the algebras of polynomial differential operators in even or odd variables. Similarly, the standard modules $\mathcal{P}_{\mathbf{h}}(n)$ and $\mathcal{G}_{\mathbf{h}}(n)$ over the rings of $\mathbf{h}$-differential operators can be regarded as deformations of the polynomial rings $\mathcal{P}(n)$ and $\mathcal{G}(n)$ respectively.

Our next task is the calculation of the contravariant form on standard modules $\mathcal{P}_{\mathbf{h}}(n)$ and their skew versions $\mathcal{G}_{\mathbf{h}}(n)$. To perform it we use Zhelobenko automorphisms of the double coset reduction algebra, see Section 2.2. First we establish a covariance property of the contravariant form with respect to these automorphisms, see Section 3.4. Next, following [5], we use the connection of the contravariant form on the reduction algebras $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ and $\mathcal{G}\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ to the Zhelobenko automorphism $\check{\xi}_{w_0}$ where w_0 is the longest element of the Weyl group of $\mathbf{gl}_n$. The origin of this connection goes back to Zhelobenko, see [6, Chapter 5]. In [5], this connection was used for the proof of irreducibility of the images of intertwining operators between certain standard modules of the Yangians. We present a slightly different from [5] proof of this connection and then compute the contravariant form on polynomial representations of the algebras $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ and $\mathcal{G}\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ in two ways: first, with the help of $\check{\xi}_{w_0}$ and, second, by direct computations in the latter reduction algebras.

The specialization of the $\bar{\mathrm{U}}(\mathbf{h})$-valued contravariant form to a generic weight λ coincides with a restriction of the $\mathbf{gl}_n$-contravariant form to singular (=$\mathbf{n}_+$-invariant) vectors in tensor products $\mathcal{P}(n)\otimes M_\lambda$, respectively, $\mathcal{G}(n)\otimes M_\lambda$. This coincidence occurs as well for the tensor products $\mathcal{P}(n)\otimes L_\lambda$, respectively, $\mathcal{G}(n)\otimes L_\lambda$ where L_λ is the irreducible $\mathbf{gl}_n$-module with a dominant weight λ. In this case the contravariant form also admits a specialization. One of the main results of our paper consists in showing that these specializations of $\mathcal{P}_{\mathbf{h}}(n)$ and $\mathcal{G}_{\mathbf{h}}(n)$ present zero singular vectors iff they are in the kernel of the specialized form.

The paper is organized as follows. In Sections 2.1–2.2 we recall the definition of the Mickelsson reduction algebras $\bar{\mathrm{Z}}_\pm$ and their localization $\bar{\mathrm{Z}}$, introduce Zhelobenko automorphisms and describe in Section 2.3 our basic examples — reduction algebras $\bar{\mathcal{D}}(\mathbf{g})$, $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ and $\mathcal{G}\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$. In Section 3.3 we introduce a natural class of $(\bar{\mathrm{Z}}, \mathbf{h})$-modules over reduction algebras and a notion of $\bar{\mathrm{U}}(\mathbf{h})$-valued contravariant forms on them. We establish a connection of these forms with the contravariant forms on $\mathbf{n}_+$-invariants and $\mathbf{n}_-$-coinvariants of certain $\mathbf{g}$-modules. Here $\mathbf{g}$ is a reductive Lie algebra, $\mathbf{n}_\pm$ are their opposite nilpotent subalgebras. In Section 3.4 we describe analogues of the Harish-Chandra map for our basic examples of reduction algebras and define with their help contravariant forms on these algebras. Section 3.5 is devoted to the calculation of these forms on basic polynomial representations of the algebras $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ and $\mathcal{G}\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$. Sections 4.1–4.2 are devoted to the justification of the evaluations of the computed contravariant forms and their use for the norms of $\mathbf{n}_+$-invariant vectors in tensor products of irreducible finite-dimensional representations of the Lie algebra $\mathbf{gl}_n$ and symmetric or exterior powers of its fundamental representation. As an illustration we check in Section 4.3 that the Pieri rules follow from our calculations. Appendices contain an alternative derivation of norms of $\mathbf{n}_+$-invariant vectors.

2. Reduction algebras

2.1. Three types of reduction algebras

Let $\mathbf{g}$ be a finite-dimensional reductive Lie algebra with a fixed triangular decomposition $\mathbf{g} = \mathbf{n}_+ + \mathbf{h} + \mathbf{n}_-$, where $\mathbf{h}$ is Cartan subalgebra, $\mathbf{n}_+$ and $\mathbf{n}_-$ are two opposite nilpotent subalgebras. We denote by Δ the root system of $\mathbf{g}$ and by Δ_+ the set of positive roots. Let $\mathcal{A}$ be an associative algebra which contains the universal enveloping algebra $\mathrm{U}(\mathbf{g})$. In particular, $\mathcal{A}$ is a $\mathrm{U}(\mathbf{g})$-bimodule with respect to the left and right multiplications by elements of $\mathrm{U}(\mathbf{g})$. We assume that $\mathcal{A}$ is free as the left $\mathrm{U}(\mathbf{g})$-module and, moreover, that $\mathcal{A}$ contains a subspace V, invariant with respect to the adjoint action of $\mathrm{U}(\mathbf{g})$ such that $\mathcal{A}$ is isomorphic to $\mathrm{U}(\mathbf{g}) \otimes V$ as the left $\mathrm{U}(\mathbf{g})$ module. The action on $\mathrm{U}(\mathbf{g}) \otimes V$ is diagonal. The adjoint action of $\mathbf{g}$ on V is assumed to be reductive.

In this setting we have three natural reduction algebras. The Mickelsson [7] reduction algebra $\mathrm{Z}_+ = \mathrm{Z}(\mathcal{A}, \mathbf{n}_+)$ is defined as the quotient of the normalizer of the left ideal $\mathrm{J}_+ = \mathcal{A}\mathbf{n}_+$ modulo J_+. The Mickelsson reduction algebra $\mathrm{Z}_- = \mathrm{Z}(\mathcal{A}, \mathbf{n}_-)$ is defined as the quotient of the normalizer of the right ideal $\mathrm{J}_- = \mathbf{n}_-\mathcal{A}$ modulo J_-.

In the following we assume that $\mathcal{A}$ is equipped with an anti-involution ε whose restriction to $\mathrm{U}(\mathbf{g})$ coincides with the Cartan anti-involution:

$$\varepsilon(e_{\alpha_c}) = e_{-\alpha_c}, \qquad \varepsilon(h) = h \text{ for any } h \in \mathbf{h}, \tag{1}$$