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Chern–Osserman type equality for complete surfaces in $\mathbb{R}^n$

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ABSTRACT

We obtain a Chern–Osserman type equality of a complete immersed surface in Euclidean space, provided the L^2-norm of the second fundamental form is finite. Also, by using a monotonicity formula, we prove that if the L^2-norm of mean curvature of a noncompact surface is finite, then it has at least quadratic area growth.

1. Introduction

Let M be a complete noncompact and connected 2-dimensional Riemannian manifold with finite total Gaussian curvature, the classical result due to Huber [1] says that M is of finite topological type, namely, M is homeomorphic to a compact surface with finite number of points punctured; Cohn-Vossen [2] proved the following inequality,

$$\chi(M) - \frac{1}{2\pi}\int_M K\, d\sigma \geq 0, \tag{1.1}$$

where K is the Gaussian curvature of M, $\chi(M)$ is the Euler characteristic of M. The explicit formula was obtained by Shiohama [3] that

$$\chi(M) - \frac{1}{2\pi}\int_M K\, d\sigma = \lim_{t\to\infty}\frac{area(D(t))}{\pi t^2}, \tag{1.2}$$

where $D(t)$ denote the geodesic balls of M of radius t at a fixed point.

If M is a complete minimal surface in $\mathbb{R}^n$ with finite total Gaussian curvature, Chern and Osserman [4,5] established the following inequality

$$-\chi(M) \leq -\frac{1}{2\pi}\int_M Kd\sigma - k(M), \tag{1.3}$$

where $k(M)$ is the number of ends of M. The explicit equality was obtained by Jorge and Meeks [6] that

$$-\chi(M) = -\frac{1}{2\pi}\int_M Kd\sigma - \lim_{t\to\infty}\frac{area(M\cap B(t))}{\pi t^2}, \tag{1.4}$$

where $B(t)$ is the geodesic ball of $\mathbb{R}^n$ with radius t.

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In this paper we study the complete noncompact immersed surfaces in $\mathbb{R}^n$ with finite total extrinsic curvature, and obtained an analogous formula of (1.4). Our main theorem is

Theorem 1.1. *Let M be an oriented, connected, complete and noncompact surface immersed in $\mathbb{R}^n$, A the second fundamental form of the immersion, K the Gaussian curvature, $\chi(M)$ the Euler characteristic of M, $B(t)$ the geodesic ball of $\mathbb{R}^n$ from a fixed point with radius t. Suppose $\int_M |A|^2 d\sigma < \infty$, then $\lim_{t\to\infty} \frac{area(M\cap B(t))}{\pi t^2}$ is a positive integer and*

$$\lim_{t\to\infty} \frac{area(M\cap B(t))}{\pi t^2} = \chi(M) - \frac{1}{2\pi}\int_M K d\sigma .$$

Remark 1.1. If M is minimally immersed, by the Gaussian equation $K = -\frac{1}{2}|A|^2$, then the finiteness of total Gaussian curvature would imply the finiteness of total extrinsic curvature.

With the condition $\int_M |A|^2 d\sigma < \infty$, White [7] proved that $\frac{1}{2\pi}\int_M K d\sigma$ must be an integer, and Muller–Sverak [8] proved that M is properly immersed. On the other hand, by the Gaussian equation, $\int_M |A|^2 d\sigma < \infty$ implies that $\int_M |K| d\sigma < \infty$. Thus under the assumption of Theorem 1.1, classical results of Huber [1], Cohn-Vossen [2] and Shiohama [3] are all valid. More precisely, choose a point of M as the origin, then intrinsic ball of M is subset of extrinsic ball of M, in the same radius. Therefore we have

$$area(M\cap B(t)) \ge area(D(t)).$$

Shiohama's formula (1.2) then implies

$$\chi(M) - \frac{1}{2\pi}\int_M K d\sigma \le \lim_{t\to\infty} \frac{area(M\cap B(t))}{\pi t^2}. \tag{1.5}$$

What we do in this paper is to prove that

$$\chi(M) - \frac{1}{2\pi}\int_M K d\sigma \ge \lim_{t\to\infty} \frac{area(M\cap B(t))}{\pi t^2}. \tag{1.6}$$

This inequality, together with White's result [7], implies that $\lim_{t\to\infty} \frac{area(M\cap B(t))}{\pi t^2}$ is an integer. The positivity of the integer is followed by Corollary 1.2.

Our proof follows the argument used in [9], where the Chern–Osserman type inequality of complete minimal surface in hyperbolic space form was obtained. Esteve–Palmer [10] obtained more general theorems for minimal surfaces in certain Cartan–Hadamard manifold. Without the assumption of minimal immersion, we first establish two monotonicity formulas for functions consisting of area of extrinsic ball and certain norms of mean curvature (Theorem 2.4), then we derive the inequality (1.6) by careful calculation.

The monotonicity formulas also have an interesting application, namely, if the L^2-norm of mean curvature H of the surface is finite, then it has at least quadratic area growth.

Corollary 1.2 (*See also* Corollary 2.5). *Let M be a complete properly immersed noncompact surface in $\mathbb{R}^n$ with $\int_M |H|^2 d\sigma < \infty$, then the area of the extrinsic balls of M has at least quadratic area growth.*

2. Preliminaries

Let $x : M \to \mathbb{R}^n$ be a complete properly immersed surface in $\mathbb{R}^n$, r the distance function of $\mathbb{R}^n$ from a fixed point. For simplicity, we always assume the fixed point to be 0, unless otherwise specified. Denote the covariant derivative of $\mathbb{R}^n$ and M by $\overline{\nabla}$ and ∇ respectively. Let X, Y be two tangent vector fields of M, then

$$\begin{aligned}(\overline{\nabla}^2 r)(X, Y) &= XY(r) - \overline{\nabla}_X Y(r)\\ &= (\nabla^2 r)(X, Y) - \langle A(X, Y), \overline{\nabla} r\rangle.\end{aligned} \tag{2.1}$$

The equality (2.1), together with the fact that $\overline{\nabla}^2 r = \frac{1}{r}(g_{st} - dr\otimes dr)$, where g_{st} denotes the standard metric of $\mathbb{R}^n$, implies

Proposition 2.1. *For any unit tangent vector e of M,*

$$(\nabla^2 r)(e, e) = \frac{1}{r}(1 - \langle e, \nabla r\rangle^2) + \langle A(e, e), \nabla^\perp r\rangle,$$

where $\nabla^\perp r$ is the projection of $\overline{\nabla} r$ onto the normal of M.

By Sard's theorem, for *a.e.* $t > 0$, $M_t = \{x \in M : r(x) < t\}$ is a related compact open subset of M with the boundary ∂M_t being a closed immersed curve of M. Let $v(t) = areaM_t$, A the second fundamental form of M, and $H = trA$ the mean curvature vector.