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Vector bundles for "Matrix algebras converge to the sphere"

Marc A. Rieffel

Department of Mathematics, University of California, Berkeley, CA 94720-3840, United States

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ABSTRACT

In the high-energy quantum-physics literature, one finds statements such as "matrix algebras converge to the sphere". Earlier I provided a general precise setting for understanding such statements, in which the matrix algebras are viewed as quantum metric spaces, and convergence is with respect to a quantum Gromov–Hausdorff-type distance.

But physicists want even more to treat structures on spheres (and other spaces), such as vector bundles, Yang–Mills functionals, Dirac operators, etc., and they want to approximate these by corresponding structures on matrix algebras. In the present paper we treat this idea for vector bundles. We develop a general precise way for understanding how, for two compact quantum metric spaces that are close together, to a given vector bundle on one of them there can correspond in a natural way a unique vector bundle on the other. We then show explicitly how this works for the case of matrix algebras converging to the 2-sphere.

1. Introduction

In several earlier papers [1–4] I showed how to give a precise meaning to statements in the literature of high-energy physics and string theory of the kind "matrix algebras converge to the sphere". (See the references to the quantum physics literature given in [1,5–10].) I did this by introducing and developing a concept of "compact quantum metric spaces", and a corresponding quantum Gromov–Hausdorff-type distance between them. The compact quantum spaces are unital C*-algebras, and the metric data is given by equipping the algebras with suitable seminorms that play the role of the usual Lipschitz seminorms on the algebras of continuous functions on ordinary compact metric spaces. The natural setting for "matrix algebras converge to the sphere" is that of coadjoint orbits of compact semi-simple Lie groups, as shown in [1,3,4].

But physicists need much more than just the algebras. They need vector bundles, gauge fields, Dirac operators, etc. In the present paper I provide a general method for giving precise quantitative meaning to statements in the physics literature of the kind "here are the vector bundles over the matrix algebras that correspond to the monopole bundles on the sphere" [11–20,6]. I then apply this method to the case of the 2-sphere, with full proofs of convergence. Many of the considerations in this paper apply directly to the general case of coadjoint orbits. But some of the detailed estimates needed to prove convergence require fairly complicated considerations (see Section 11) concerning highest weights for representations of compact semi-simple Lie groups. It appears to me that it would be quite challenging to carry out those details for the

E-mail address: rieffel@math.berkeley.edu.

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general case, though I expect that some restricted cases, such as matrix-algebra approximations for complex projective spaces [21,14,22–24], are quite feasible to deal with.

In [5] I studied the convergence of ordinary vector bundles on ordinary compact metric spaces for the ordinary Gromov–Hausdorff distance. The approach that worked for me was to use the correspondence between vector bundles and projective modules (Swan's theorem [25]), and by this means represent vector bundles by corresponding projections in matrix algebras over the algebras of continuous functions on the compact metric spaces; and then to prove appropriate convergence of the projections. In the present paper we follow that same approach, in which now we also consider projective modules over the matrix algebras that converge to the 2-sphere, and thus also projections in matrix algebras over these matrix algebras.

For this purpose, one needs Lipschitz-type seminorms on all of the matrix algebras over the underlying algebras, with these seminorms coherent in the sense that they form a "matrix seminorm". In my recent paper [4] the theory of these matrix seminorms was developed, and properties of such matrix seminorms for the setting of coadjoint orbits were obtained. In particular, some general methods were given for obtaining estimates related to how these matrix seminorms mesh with an appropriate quantum analog of the Gromov–Hausdorff distance. The results of that paper will be used here.

Recently Latrémolière introduced an improved version of quantum Gromov–Hausdorff distance [26], which he calls "quantum Gromov–Hausdorff propinquity". In [4] I showed that his propinquity works very well for our setting of coadjoint orbits, and so propinquity is the form of quantum Gromov–Hausdorff distance that we use in the present paper. Latrémolière defines his propinquity in terms of an improved version of the "bridges" that I had used in my earlier papers. For our matrix seminorms we need corresponding "matricial bridges". In [4] natural such matricial bridges were constructed for the setting of coadjoint orbits. They will be used here.

To give somewhat more indication of the nature of our approach, we give now an imprecise version of one of the general theorems that we apply. For simplicity of notation, we express it here only for the C*-algebras, rather than for matrix algebras over the C*-algebras as needed later. Let $(\mathcal{A}, L^{\mathcal{A}})$ and $(\mathcal{B}, L^{\mathcal{B}})$ be compact quantum metric spaces, where $\mathcal{A}$ and $\mathcal{B}$ are unital C*-algebras and $L^{\mathcal{A}}$ and $L^{\mathcal{B}}$ are suitable seminorms on them. Let Π be a bridge between $\mathcal{A}$ and $\mathcal{B}$. Then $L^{\mathcal{A}}$ and $L^{\mathcal{B}}$ can be used to measure Π. We denote the resulting length of Π by l_Π. Then we will see, imprecisely speaking, that Π together with $L^{\mathcal{A}}$ and $L^{\mathcal{B}}$ determine a suitable seminorm, L_Π, on $\mathcal{A} \oplus \mathcal{B}$.

Theorem 1.1 (*Imprecise Version of Theorem 5.7*). *Let $(\mathcal{A}, L^{\mathcal{A}})$ and $(\mathcal{B}, L^{\mathcal{B}})$ be compact quantum metric spaces, and let Π be a bridge between $\mathcal{A}$ and $\mathcal{B}$, with corresponding seminorm L_Π on $\mathcal{A} \oplus \mathcal{B}$. Let l_Π be the length of Π as measured using $L^{\mathcal{A}}$ and $L^{\mathcal{B}}$. Let $p \in \mathcal{A}$ and $q \in \mathcal{B}$ be projections. If $l_\Pi L_\Pi(p, q) < 1/2$, and if q_1 is another projection in $\mathcal{B}$ such that $l_\Pi L_\Pi(p, q_1) < 1/2$, then there is a continuous path, $t \to q_t$ of projections in $\mathcal{B}$ going from q to q_1, so that the projective modules corresponding to q and q_1 are isomorphic. In this way, to the projective $\mathcal{A}$-module determined by p we have associated a uniquely determined isomorphism class of projective $\mathcal{B}$-modules.*

In Sections 7 through 12, theorems of this type are then applied to the specific situation of matrix algebras converging to the 2-sphere, in order to obtain our correspondence between projective modules over the 2-sphere and projective modules over the matrix algebras.

Very recently Latrémolière introduced a fairly different way of saying when two projective modules over compact quantum metric spaces that are close together correspond [27]. For this purpose he equips projective modules with seminorms that play the role of a weak analog of a connection on a vector bundle over a Riemannian manifold, much as our seminorms on a C*-algebra are a weak analog of the total derivative (or of the Dirac operator) of a Riemannian manifold. As experience is gained with more examples it will be interesting to discover the relative strengths and weaknesses of these two approaches.

My next project is to try to understand how the Dirac operator on the 2-sphere is related to "Dirac operators" on the matrix algebras that converge to the 2-sphere, especially since in the quantum physics literature there are at least three inequivalent Dirac operators suggested for the matrix algebras. This will involve results from [28].

2. Matrix Lip-norms and state spaces

As indicated above, the projections representing projective modules are elements of matrix algebras over the basic C*-algebras. In this section we give some useful perspective on the relations between certain types of seminorms on matrix algebras over a given C*-algebra and the metrics on the state spaces of the matrix algebras that come from the seminorms.

Let $\mathcal{A}$ be a unital C*-algebra. For a given natural number d let M_d denote the algebra of $d \times d$ matrices with entries in $\mathbb{C}$, and let $M_d(\mathcal{A})$ denote the C*-algebra of $d \times d$ matrices with entries in $\mathcal{A}$. Thus $M_d(\mathcal{A}) \cong M_d \otimes \mathcal{A}$. Since $\mathcal{A}$ is unital, we can, and will, identify M_d with the subalgebra $M_d \otimes 1_{\mathcal{A}}$ of $M_d(\mathcal{A})$.

We recall from definition 2.1 of [4] that by a "slip-norm" on a unital C*-algebra $\mathcal{A}$ we mean a $*$-seminorm L on $\mathcal{A}$ that is permitted to take the value $+\infty$ and is such that $L(1_A) = 0$. Given a slip-norm $L^{\mathcal{A}}$ on $\mathcal{A}$, we will need slip-norms, $L_d^{\mathcal{A}}$, on each $M_d(\mathcal{A})$ that correspond somewhat to $L^{\mathcal{A}}$. It is reasonable to want these seminorms to be coherent in some sense as d varies. As discussed before definition 5.1 of [4], the appropriate coherence requirement is that the sequence $\{L_d^{\mathcal{A}}\}$ forms a "matrix slip-norm". To recall what this means, for any positive integers m and n we let M_{mn} denote the linear space of $m \times n$ matrices with complex entries, equipped with the norm obtained by viewing such matrices as operators from the Hilbert space $\mathbb{C}^n$ to the Hilbert space $\mathbb{C}^m$. We then note that for any $A \in M_n(\mathcal{A})$, for any $\alpha \in M_{mn}$, and any $\beta \in M_{nm}$, the usual matrix product $\alpha A \beta$ is in $M_m(\mathcal{A})$.