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Nonlinear waves in layered media: Solutions of the KdV–Burgers equation

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ABSTRACT

We use the KdV–Burgers equation to model a behaviour of a soliton which, while moving in non-dissipative medium encounters a barrier with dissipation. The modelling included the case of a finite width dissipative layer as well as a wave passing from a non-dissipative layer into a dissipative one.

The dissipation results in reducing the soliton amplitude/velocity, and a reflection and refraction occur at the boundary(s) of a dissipative layer. In the case of a finite width barrier on the soliton path, after the wave leaves the dissipative barrier it retains a soliton form and a reflection wave arises as small and quasi-harmonic oscillations (a breather). The first order approximation in the expansion by the small dissipation parameter is studied.

0. Introduction

The generalized KdV–Burgers (KdV–B) equation considered here is of the form

$$u_t(x,t) = \gamma(x)u_{xx}(x,t) + 2u(x,t)u_x(x,t) + u_{xxx}(x,t). \tag{1}$$

It is related to the viscous and dispersive medium. The viscosity dampens oscillations except for stationary (or travelling wave) solutions.

Note that $\gamma(x) = 0$ corresponds to the KdV equations whose travelling waves solutions are solitons and $\gamma(x) = \text{const} > 0$ corresponds to the KdV–Burgers equation whose travelling waves solutions are shock waves.

The layered media consist of layers with both dispersion an dissipation and layers without dissipation. In the latter case the waves are described by the KdV equation, while in the former — by the KdV–Burgers one. Thus we consider three possibilities for $\gamma(x)$.

1. $\gamma(x) = \alpha(1-\theta(x))$ is the Heaviside step function(the two-layer case);
2. $\gamma(x) = \alpha(\theta(x-\beta) - \theta(x+\beta))$ is a Π-form density of viscosity (the three-layer case);
3. $\gamma(x) = \alpha \operatorname{sech}^2(\beta x)$ is a function with (numerically) compact support (the three-layer case).

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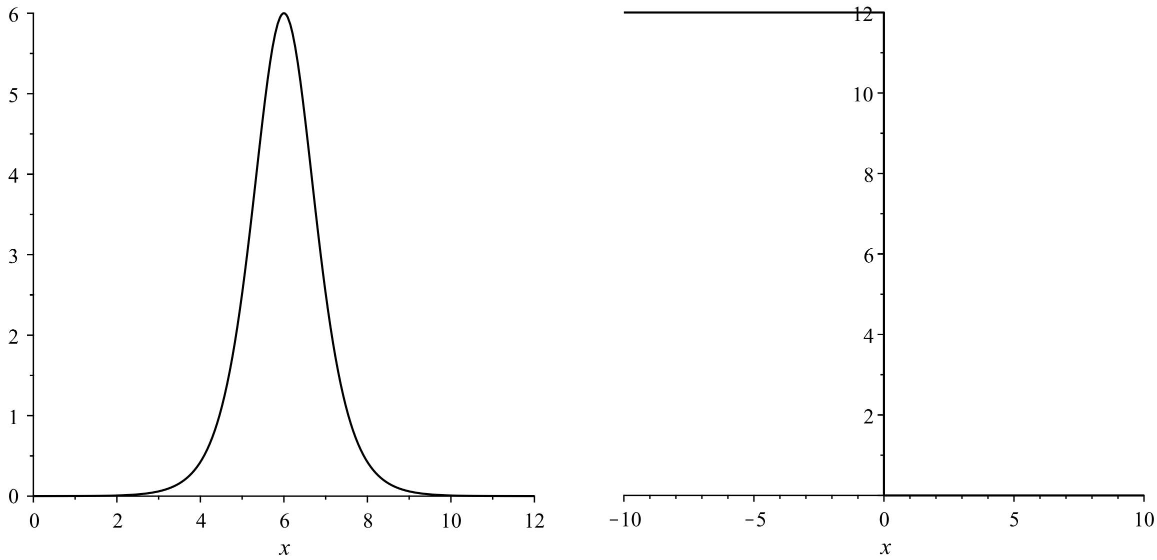

Fig. 1. Left: Soliton $u(x, t, a, s) = 6a^2\text{sech}^2(4a^3t+a(x+s))$. Velocity $V = -4a^2 = 4$, initial shift $s = -6$. Right: Step-like obstacle $\gamma(x) = \alpha(1-\theta(x)),\ \alpha = 6$.

The solution entering the layer with $\gamma(x) \neq 0$ obeys the KdV–B equation. The situation is similar to the geometric optics: as a ray enters water from the air, one can observe the reflected wave and a decay of the transient wave.

We use the following initial value– boundary problem (IVBP) for the KdV–Burgers equation on $\mathbb{R}$:

$$u(x, 0) = u_0(x, a, s) = 6a^2\text{sech}^2(a(x + s)),\ \ u(\pm\infty, t) = 0,\ \ u_x(\pm\infty, t) = 0. \tag{2}$$

For numerical computations we use $x \in [a, b]$ for appropriately large $a,\ b$ instead of $\mathbb{R}$. Note that $u_0(x, a, s) = 6a^2\text{sech}^2(a(x+s)) = 6a^2\text{sech}^2(4a^3t + a(x + s))\big|_{t=0}$ is an initial placement of a soliton moving to the left, see Fig. 1 (left).

This paper is a continuation of our study [1]. We use the KdV–B equation to model a behaviour of the soliton which, while moving in non-dissipative medium encounters a barrier with dissipation. The modelling included the case of a finite width dissipative layer as well as a wave passing from a non-dissipative layer into a dissipative one.

The dissipation results in reducing the soliton amplitude/velocity, and a reflection and refraction occur at the boundary(s) of a dissipative layer. In the case of a finite width barrier on the soliton path, after the wave leaves the dissipative barrier it retains a soliton form, yet a reflection wave arises as small and quasi-harmonic oscillations (a breather), see [2–4] for detail of such effects. We use the first-order approximation in the expansion by the small dissipation parameter to obtain the breather description. Some estimation of a refraction on a barrier is also obtained.

1. Soliton in 2-layer medium

This case models a passage from non-dissipative half-space to a dissipative one. We take $\gamma(x) = \alpha(1-\theta(x))$ as a dissipation distribution function to present a single boundary separating these half-spaces, see Fig. 1 (right).

It is natural to expect each solution to behave as the one of the KdV at the right half-space and as solution of KdV–B at the left one.

The process of transition from the soliton to the correspondent solution of the KdV–B is predictable. The transient wave in a dissipative media becomes a solitary shock which loses speed and decays to become nonexistent at $t \to +\infty$; and a reflected wave is seen in the non-dissipative half-space, see Fig. 2 for result of numeric modelling.

2. Three layers

This case models a dissipative barrier between two non-dissipative half-spaces. A soliton solution of the KdV equation, meeting a layer with dissipation, transforms somewhat similarly to a ray of light in the air crossing a semi-transparent plate. We model a barrier by $\gamma(x) = \alpha(\theta(x - \beta) - \theta(x + \beta))$ (a Π-form density with a compact support, Fig. 3 (left)) or by $\gamma(x) = a\text{sech}^2(ax)$ (Fig. 6 (left)), which is negligibly small far from origin.

It is natural to expect each solution to behave as one of the KdV outside the vicinity of the origin and as a solution of KdV–B near the origin.