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Comment on the Bekenstein bound

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ABSTRACT

We propose a rigorous derivation of the Bekenstein upper limit for the entropy/information that can be contained by a physical system in a given finite region of space with given finite energy. The starting point is the observation that the derivation of such a bound provided by Casini (2008) is similar to the description of the black hole incremental free energy that had been given in Longo (1997). The approach here is different but close in the spirit to Casini (2008). Our bound is obtained by operator algebraic methods, in particular Connes' bimodules, Tomita–Takesaki modular theory and Jones' index are essential ingredients inasmuch as the von Neumann algebras in question are typically of type *III*. We rely on the general mathematical framework, recently set up in Longo (2018), concerning quantum information of infinite systems.

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1. Introduction

The Bekenstein bound is a universal limit on the entropy that can be contained in a physical system of given size and total energy [1]. If a system of total energy E, including rest mass, is enclosed in a sphere of radius R, then the entropy S of the system is bounded by

$$S \leq \lambda RE\,,$$

where $\lambda > 0$ is a constant (the value $\lambda = 2\pi$ is often proposed).

In [2], H. Casini gave an interesting derivation for this bound, based on relative entropy considerations. It was observed, following [3], that, in order to get a finite measure for the entropy carried by the system in a region of the space, one should subtract from the bare entropy of the local state the entropy corresponding to the vacuum fluctuations, which is entirely due to the localization. A similar subtraction can be done to define a localized form of energy.

The argument in [2] is following. One considers a space region V and the von Neumann algebra $\mathcal{A}(O)$ of the observables localized in the causal envelop O of V. The restriction ρ_V of a global state ρ to $\mathcal{A}(O)$ has formally an entropy given by von Neumann's entropy

$$S(\rho_V) = -\,\mathrm{Tr}(\rho_V \log \rho_V)\,,$$

that is known to be infinite. So one subtracts the vacuum state entropy

$$S_V = S(\rho_V) - S(\rho^0_V)$$

with ρ^0_V the density matrix of the restriction of the vacuum state ρ^0 to $\mathcal{A}(O)$.

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Similarly, if K is the Hamiltonian for V, then one considers the difference of the expectations of K in the given state and in the vacuum state

$$K_V = \mathrm{Tr}(\rho_V K) - \mathrm{Tr}(\rho_V^0 K)\,. \tag{1}$$

The version of the Bekenstein bound in [2] is $S_V \leq K_V$, namely

$$S(\rho_V) - S(\rho_V^0) \leq \mathrm{Tr}(\rho_V K) - \mathrm{Tr}(\rho_V^0 K) \tag{2}$$

which is equivalent to

$$S(\rho_V|\rho_V^0) \equiv \mathrm{Tr}\big(\rho_V(\log\rho_V - \log\rho_V^0)\big) \geq 0\,,$$

namely to the positivity of the relative entropy. One is then left to estimate the right hand side of (2). Here the (dimensionless) local Hamiltonian K is defined by $\rho_V^0 = e^{-K}/\mathrm{Tr}(e^K)$, up to a scalar shifting that does not affect the definition of (1).

The above argument, thought in terms of a cutoff theory, breaks for general Quantum Field Theory as the local von Neumann algebras $\mathcal{A}(O)$ are not of type I; under general assumptions, $\mathcal{A}(O)$ is a factor of type III so no trace Tr and no density matrix ρ are definable. Yet, as is well known, modular theory and Araki's relative entropy $S(\varphi|\psi)$ are definable in general. We aim at a different argument, close in the spirit to the above discussion, that makes rigorous sense.

The point is that the above argument is quite similar to the rigorous description of the black hole incremental free energy and entropy given in [4] in the general Quantum Field Theory framework. Recently, this work led to a universal formula for the incremental free energy, that can be interpreted in several different contexts. This paper is an illustration of this fact.

We take the point of view that relative entropy is a primary concept and other entropy quantities should be expressed in terms of relative entropies (cf. also [5]). This is the case, for example, for the von Neumann entropy. The von Neumann entropy $S(\varphi)$ of a state φ of a von Neumann algebra $\mathcal{M}$ may be expressed in terms of the relative entropy:

$$S(\varphi) = \sup_{(\varphi_i)} \sum_i S(\varphi|\varphi_i)$$

where the supremum is taken over all finite families of positive linear functionals φ_i of $\mathcal{M}$ with $\sum_i \varphi_i = \varphi$ (see [6]). Clearly $S(\varphi)$ cannot be finite unless $\mathcal{M}$ is of type I.

However, rather than tracing back the Bekenstein bound to the positivity of the relative entropy, here we are going to rely on the positivity of the incremental free energy, or conditional entropy, which can be obtained in two possible ways: by the monotonicity of the relative entropy in relations to Connes–Stø rmer's entropy [7], or by linking it to Jones' index [8]. In this respect, our argument is close to the derivation of the bound in [9], that relies on the monotonicity of the relative entropy.

2. Bound for the entropy

We now are going to compare two states of a physical system, ω_{in} is a suitable reference state, e.g. the vacuum in QFT, and ω_{out} is a state that can be reached from ω_{in} by some physically realizable process (quantum channel). We relate the incremental energy and the entropy.

2.1. Mathematical and general setting

With $\mathcal{N}, \mathcal{M}$ being von Neumann algebras, an $\mathcal{N}-\mathcal{M}$ bimodule is a Hilbert space $\mathcal{H}$ equipped with a normal representation ℓ of $\mathcal{N}$ on $\mathcal{H}$ and a normal anti-representation r of $\mathcal{M}$ on $\mathcal{H}$, and $\ell(n)$ commutes with $r(m)$, for all $n \in \mathcal{N}$, $m \in \mathcal{M}$ [10]. For simplicity, here we assume that $\mathcal{N}$ and $\mathcal{M}$ are factors. A vector $\xi \in \mathcal{H}$ is said to be cyclic for $\mathcal{H}$ if it is cyclic for the von Neumann algebra $\ell(\mathcal{N}) \vee r(\mathcal{M})$ generated by $\ell(\mathcal{N})$ and $r(\mathcal{M})$.

Proposition 2.1. *Let $\alpha : \mathcal{N} \to \mathcal{M}$ be a completely positive, normal, unital map and ω a faithful normal state of $\mathcal{M}$. Then there exists an $\mathcal{N}-\mathcal{M}$ bimodule $\mathcal{H}_\alpha$, with a cyclic vector $\xi_\alpha \in \mathcal{H}$ and left and right actions ℓ_α and r_α, such that*

$$(\xi_\alpha, \ell_\alpha(n)\xi_\alpha) = \omega_{\text{out}}(n)\,, \quad (\xi_\alpha, r_\alpha(m)\xi_\alpha) = \omega_{\text{in}}(m)\,, \tag{3}$$

with $\omega_{\text{in}} \equiv \omega$, $\omega_{\text{out}} \equiv \omega_{\text{in}} \cdot \alpha$. The pair $(\mathcal{H}_\alpha, \xi_\alpha)$ with this property is unique up to unitary equivalence.

Conversely, given an $\mathcal{N}-\mathcal{M}$ bimodule $\mathcal{H}$ with a cyclic vector $\xi \in \mathcal{H}$, with $\omega = (\xi, r(\cdot)\xi)$ faithful state of $\mathcal{M}$, there is a unique completely positive, unital, normal map $\alpha : \mathcal{N} \to \mathcal{M}$ such that $(\mathcal{H}, \xi) = (\mathcal{H}_\alpha, \xi_\alpha)$, the cyclic bimodule associated with α by ω.

Proof. For the construction of $(\mathcal{H}_\alpha, \xi_\alpha)$, let $\mathcal{M}$ act on a Hilbert space with cyclic and separating vector ξ such that $\omega(m) = (\xi, m\xi)$. The GNS representation of the algebraic tensor product $\mathcal{N} \odot \mathcal{M}^o$ ($\mathcal{M}^o$ the opposite algebra of $\mathcal{M}$), associated with the state determined by

$$n \odot m^o \mapsto (\xi, \alpha(n)J_{\mathcal{M}}m^*J_{\mathcal{M}}\xi) \tag{4}$$

gives $(\mathcal{H}_\alpha, \xi_\alpha)$, see [11].

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