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An analytical approach to the external force-free motion of pendulums on surfaces of constant curvature

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ABSTRACT

The dynamics of external force free motion of pendulums on surfaces of constant Gaussian curvature is addressed when the pivot moves along a geodesic obtaining the Lagrangian of the system. As an application it is possible the study of elastic and quantum pendulums.

1. Introduction

The classical theory of force-free motions of rigid particle systems has a long history in connection with investigation in Mathematics and Mechanics, from Differential Calculus, in the case of translational particle motions in Euclidean space (Newton axioms of Mechanics), to the Riemann Geometry. Hence, the definition of geodesics has immediately concerned with force-free particle motions on surfaces. The introduction of Riemannian manifolds and the geometry of their geodesics were motivated by the mechanics of constrained particle systems. In the last few years, the study of force-free motion systems on Riemannian manifolds of constant sectional curvature has attracted the interest of several authors [1–5] and [6].

In particular, in [2] the authors define the notion of a pendulum on a surface of constant Gaussian curvature K and they study the motion of a mass at a fixed distance from a pivot. So, a pendulum problem on a surface of constant curvature is defined as a pivot point and a mass connected to that point by a rigid massless rod of fixed length ρ. It is assumed that the pivot is constrained to move along some fixed curve with prescribed motion. The rod provides the only force on the mass in order to keep the mass at the fixed distance from the pivot. No torque is applied to the rod, (Fig. 1). In [2] it is studied the pendulum problem when the pivot moves along a geodesic path, and the space is a surface of constant (not zero) curvature. It is considered the surface immersed in $\mathbb{R}^3$ or $\mathbb{L}^3$ and it is obtained the differential motion equation by Newtonian procedure doing a laborious calculation. Moreover, the cases $K > 0$ and $K < 0$ are necessary to come from different forms.

In this work, we deal with the pendulum problem from an analytic point of view. This procedure allows approach simultaneously the cases of positive and negative curvature and also of zero curvature (as a limit case), with very simple

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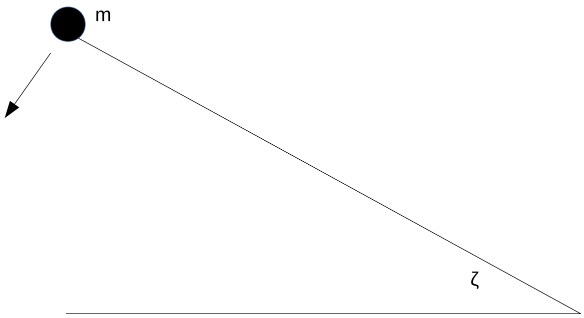

Fig. 1. Pendulum motion.

computations. The key of our study is a Lagrangian approach, which has a 1-dimensional configuration space. As a direct consequence the internal curvature force is conservative and its potential function is easily calculated. Moreover, this method can be used to another related problems, as the elastic pendulum, or the quantum pendulum.

Mainly, this work concerns the case in which the pivot moves along a geodesic. Let ζ be the angle between the rod and the motion direction of the pivot. We will assume the following convention:

If the constant curvature of the space is $K > 0$,

$$\cos_K(z) = \cos(z\sqrt{K}), \quad \sin_K(z) = \sin(z\sqrt{K}), \quad \tan_K(z) = \tan(z\sqrt{K}).$$

If the constant curvature of the space is $K < 0$,

$$\cos_K(z) = \cosh(z\sqrt{-K}), \quad \sin_K(z) = \sinh(z\sqrt{-K}), \quad \tan_K(z) = \tanh(z\sqrt{-K}).$$

The main result is the following:

Suppose that the pivot of pendulum moves with constant speed v along a geodesic on a surface with constant curvature K. Let $\zeta(t)$ be the angle at the time t between the rigid rod and the direction of pivot motion. Then the Lagrangian of the system is

$$\mathcal{L}(\zeta, \dot{\zeta}) = \frac{1}{2} m\dot{\zeta}^2 \sin_K^2(\rho)\frac{1}{|K|} - \frac{1}{2} mv^2 \mathrm{sgn}(K) \sin_K^2(\rho) \sin^2(\zeta).$$

Therefore, the motion equation is

$$\frac{d^2\zeta}{dt^2} = -v^2 K \sin_K(\zeta)\cos_K(\zeta). \tag{1}$$

Using the previous result and geometric arguments, we obtain the motion equation of the system when the pivot accelerates along a geodesic (see Section 4). Further developments are discussed in Section 5.

2. Preliminaries

It is well known that a complete, simply-connected Riemannian n-manifold with constant sectional curvature is isometric to one of model spaces $\mathbb{R}^n$, $\mathbb{S}^n(R)$ or $\mathbb{H}^n(R)$ (see [7]).

Let $(\mathbb{S}^2(R), \langle\, ,\, \rangle_1)$ be the 2-dimensional sphere of radius R, endowed with the metric induced from $(\mathbb{R}^3, \langle\, ,\, \rangle)$, where

$$\langle\, ,\, \rangle = dx^2 + dy^2 + dz^2.$$

Consider the coordinate system $\{U_1, \Phi = (\varphi, \theta)\}$ in $S^2(R)$ with $U_1 = \Phi^{-1}((0, \pi) \times (0, 2\pi))$ and $\Phi^{-1}(\varphi, \theta) = (x, y, z)$, being

$$\left.\begin{array}{l} x = R\sin\varphi\cos\theta \\ y = R\sin\varphi\sin\theta \\ z = R\cos\varphi \end{array}\right\}.$$

In this coordinate system the metric is given by

$$\langle\, ,\, \rangle_1 = R^2\, d\varphi \otimes d\varphi + R^2 \sin^2\varphi\, d\theta \otimes d\theta,$$

and the non-zero Christoffel symbols are $\Gamma^{\theta}_{\varphi\theta} = \cot\varphi$ and $\Gamma^{\varphi}_{\theta\theta} = -\sin\varphi\cos\varphi$.