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A first integrability result for Miquel dynamics

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ABSTRACT

Miquel dynamics is a discrete-time dynamical system on the space of square-grid circle patterns. For biperiodic circle patterns with both periods equal to two, we show that the dynamics corresponds to translation on an elliptic curve, thus providing the first integrability result for this dynamics. The main tool is a geometric interpretation of the addition law on the normalization of binodal quartic curves.

1. Introduction

Miquel dynamics was introduced by the second author in [1], following an original idea of Richard Kenyon [2], as a discrete-time dynamical system on the space of square-grid circle patterns. It was then conjectured that for biperiodic circle patterns, Miquel dynamics belongs to the class of discrete integrable systems, which contains among others the dimer model [3] and the pentagram map [4–6]. In this article, we show that in the particular case when both periods are equal to two, Miquel dynamics corresponds, in the right coordinates, to translation on an elliptic curve. This is the first integrability result established for Miquel dynamics. An important observation we make to prove this is a simple geometric interpretation of the addition law on the normalization of algebraic curves of degree four with two nodes.

1.1. Circle patterns and Miquel dynamics

A *square grid circle pattern* (abbreviated as SGCP) is a collection of points $(S_{i,j})_{(i,j)\in\mathbb{Z}^2}$ in the plane $\mathbb{R}^2$ such that for any $(i,j)\in\mathbb{Z}^2$, the points $S_{i,j}$, $S_{i+1,j}$, $S_{i,j+1}$ and $S_{i+1,j+1}$ are pairwise distinct and concyclic, with the circle going through them denoted by $C_{i,j}$. The circles are colored in a checkerboard pattern: the circles $C_{i,j}$ with $i+j$ even (resp. odd) are colored black (resp. white). The center of the circle $C_{i,j}$ is denoted by $O_{i,j}$. We define two maps μ_w and μ_b, respectively called *white mutation* and *black mutation*, from the set of SGCPs to itself. For any SGCP S, the SGCP $T := \mu_w(S)$ is obtained as follows: for any $(i,j)\in\mathbb{Z}^2$ such that $i+j$ is even (resp. odd), $T_{i,j}$ is obtained by reflecting $S_{i,j}$ through the line $(O_{i,j}O_{i-1,j-1})$ (resp. $(O_{i-1,j}O_{i,j-1})$). It follows from Miquel's six-circles theorem [7] that T is indeed a circle pattern, with the same black circles as S but with potentially different white circles. Similarly, for any SGCP S, the SGCP $T' := \mu_b(S)$ is obtained as follows: for any $(i,j)\in\mathbb{Z}^2$ such that $i+j$ is even (resp. odd), $T'_{i,j}$ is obtained by reflecting $S_{i,j}$ through the line $(O_{i-1,j}O_{i,j-1})$ (resp. $(O_{i,j}O_{i-1,j-1})$). Each mutation is an involution. Miquel dynamics is defined as the discrete-time dynamical system obtained

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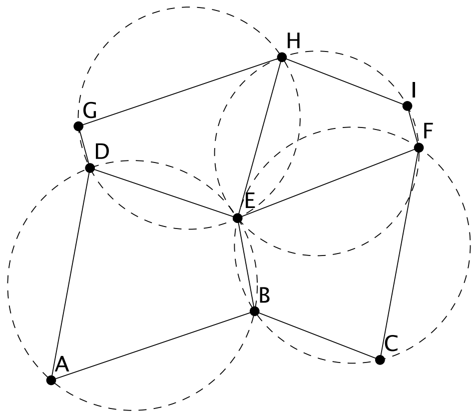

Fig. 1. Illustration of the notation for a biperiodic circle pattern with both periods equal to two.

by applying alternately μ_w followed by μ_b. Note that this dynamics is different from the one on circle configurations studied by Bazhanov, Mangazeev and Sergeev [8], which uses a different version of Miquel's theorem.

Given two positive even integers m and n and two non-collinear vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$, an SGCP S is said to be (m, n)-*biperiodic* with monodromies $\vec{u}$ and $\vec{v}$ if for any $(i, j) \in \mathbb{Z}^2$, the following two conditions hold:

1. $S_{i+m,j} = S_{i,j} + \vec{u}$;
2. $S_{i,j+n} = S_{i,j} + \vec{v}$.

We denote by $\mathcal{S}_{m,n}$ the set of all (m, n)-biperiodic SGCPs (with arbitrary monodromies). This set is stable under both black mutation and white mutation. Miquel dynamics on $\mathcal{S}_{m,n}$ is conjectured to be integrable in some sense. In this paper we provide a first integrability result in the case when $m = n = 2$. For the remainder of the paper, all SGCPs will be in $\mathcal{S}_{2,2}$.

Let $S \in \mathcal{S}_{2,2}$ be an SGCP. We will denote its vertices in the fundamental domain $\{0, 1, 2\}^2$ as follows (see Fig. 1 for an illustration):

$$\begin{array}{lll} A = S_{0,0} & B = S_{1,0} & C = S_{2,0} \\ D = S_{0,1} & E = S_{1,1} & F = S_{2,1} \\ G = S_{0,2} & H = S_{1,2} & I = S_{2,2}. \end{array}$$

Set $S_w := \mu_w(S)$ and $S_b := \mu_b(S)$. We will denote their vertices in the fundamental domain $\{0, 1, 2\}^2$ respectively by $A_w, \ldots, I_w$ and $A_b, \ldots, I_b$. Instead of looking at the absolute motion of the points, we consider the relative motion of points with respect to one another. To do so, we introduce the pattern S'_w (resp. S'_b) which is obtained from S_w (resp. S_b) by applying the translation of vector $\overrightarrow{A_w A}$ (resp. $\overrightarrow{A_b A}$). We call *renormalized white mutation* μ'_w (resp. *renormalized black mutation* μ'_b) the map which to S associates S'_w (resp. S'_b). We denote the vertices of S'_w and S'_b in the fundamental domain $\{0, 1, 2\}^2$ respectively by $A'_w, \ldots, I'_w$ and $A'_b, \ldots, I'_b$. It was shown in [1] that both points E'_w and E'_b lie on some explicit quartic curve Q_S, which also contains the points A, C, E, G and I (see Section 2 for a precise definition of Q_S). In other words, the relative motion of the point in position $(1, 1)$ with respect to the point in position $(0, 0)$ lies on this curve Q_S. The curve Q_S has, in an appropriate coordinate system, an equation of the form

$$(x^2 + y^2)^2 + ax^2 + by^2 + c = 0, \tag{1.1}$$

with $(a, b, c) \in \mathbb{R}^3$. See Fig. 2 for an example. We call a *Miquel quartic* a quartic curve which has an equation of the form (1.1). As a special case of Miquel quartics, when $a + b = 0$, we obtain the family of Cassini ovals [9].

1.2. Addition on binodal quartic curves

A complex quartic curve in $\mathbb{CP}^2$ is called *binodal* if it has two nodes, i.e. singularities at which two regular local branches intersect transversally. A binodal quartic curve is called *non-degenerate* if it has no other singularities. The projective closure in $\mathbb{CP}^2$ of a Miquel quartic is generically a non-degenerate binodal quartic curve, with its two nodes being the *circular points at infinity* (also called isotropic points) with homogeneous coordinates $(1 : \pm i : 0)$, which lie on the infinity line $\overline{\mathbb{C}}_\infty = \mathbb{CP}^2 \setminus \mathbb{C}^2$ (see also Lemma 4.2 for a more precise statement). Every non-degenerate binodal quartic has an elliptic normalization, that is a holomorphic parametrization by an elliptic curve that is bijective outside the nodes. Thus, its normalization has a natural group structure, which is unique once the neutral element is chosen. Everywhere below, whenever we write about the addition law on a non-degenerate binodal quartic, we mean the addition law on its elliptic normalization. By an abuse of