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Localization of highest weight modules of a class of extended affine Lie algebras

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ABSTRACT

In 2006, Gao and Zeng (Gao and Zeng, 2006) gave the free field realizations of highest weight modules over the extended affine Lie algebras $\widehat{\mathfrak{gl}_2(\mathbb{C}_q)}$ for any nonzero $q \in \mathbb{C}$. In the present paper, applying the technique of localization to those free field realizations, we construct a class of new weight modules with infinite dimensional weight multiplicities over $\widehat{\mathfrak{gl}_2(\mathbb{C}_q)}$. We give necessary and sufficient conditions for these modules to be irreducible. In this way, we construct free field realizations for a class of simple weight modules with infinite dimensional weight multiplicities over $\widehat{\mathfrak{gl}_2(\mathbb{C}_q)}$ for any nonzero $q \in \mathbb{C}$.

1. Introduction

In recent years, extended affine Lie algebras (EALAs) have been studied in great detail. EALAs are Lie algebras which have a non-degenerate invariant form, a self-centralizing finite dimensional ad-diagonalizable abelian subalgebra (i.e., a Cartan subalgebra), a discrete irreducible root system, and ad-nilpotency of non-isotropic root spaces (see [1,2] for definitions and structure theory). There are many EALAs which allow not only the Laurent polynomial algebras as co-ordinate algebras but also quantum tori, Jordan tori and Octonion tori as co-ordinate algebras depending on the type of algebras (see [1–5]). For instances, EALAs of type A_{d-1} are tied up with the Lie algebra $\mathfrak{gl}_d(\mathbb{C}_q)$. Quantum tori are important algebras in the theories of algebra and non-commutative geometry, see [6] and [7]. To get an EALA one has to form appropriate central extension of $\mathfrak{gl}_d(\mathbb{C}_q)$ and add certain outer derivations (just like one obtains an affine Kac–Moody Lie algebra from a loop algebra by forming a one-dimensional central extension and then adding the degree derivation). Unlike the affine Lie algebras, the representation theory of EALAs is far from well developed. See [8–16] for several interesting results on the representation theory for the extended affine Lie algebras.

Free field realizations of Lie algebras play important role in representation theory and conformal field theory. For an affine Lie algebra $\mathfrak{L}$, imaginary Verma modules of $\mathfrak{L}$ arise from non-standard partitions of the root system of $\mathfrak{L}$, see [17]. A q-version of imaginary Verma modules for the quantum groups of type $U_q(A_1^{(1)})$ was constructed in [18], and was further studied in [19,20]. Free field realizations of imaginary Verma modules over the affine Lie algebra $A_1^{(1)}$ were constructed in [21]

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for zero central charge and in [22] for arbitrary central charge. In [23], the localization technique was used to construct a new family of free field realizations. In particular, the (twisted) localization of imaginary Verma modules, for the Kac–Moody Lie algebra $A_1^{(1)}$, provides new irreducible weight dense modules with infinite weight multiplicities. In [24], the free field realizations of highest weight modules over the extended affine Lie algebra $\widehat{\mathfrak{gl}_2(\mathbb{C}_q)}$ were first given. In [25], those realizations were generalized to $\widehat{\mathfrak{gl}_d(\mathbb{C}_q)}$ and the simplicity of the corresponding modules was also given. In the present paper, inspired by the idea of [23], we will construct free field realizations for a class of simple weight modules with infinite weight multiplicities over the extended affine Lie algebras $\widehat{\mathfrak{gl}_2(\mathbb{C}_q)}$ for any nonzero $q \in \mathbb{C}$.

The organization of the paper is as follows. In Section 2, we recall the free field realizations of highest weight modules M for any nonzero $\mu \in \mathbb{C}$ and their simplicity. In Section 3.1, we collect some preliminary results on the twisted localization. Since M is a proper submodule of $\mathcal{D}_{\mathbf{m}}M$, $\mathcal{D}_{\mathbf{m}}M$ is reducible. In Section 3.2, we decide the simplicity of the quotient module $\mathcal{D}_{\mathbf{m}}M/M$. We show that $\mathcal{D}_{\mathbf{m}}M/M$ is simple if and only if $\mu \notin \mathbb{Z}$, see Theorem 3.5. In Section 3.3, we give the simplicity of $\mathcal{D}_{\mathbf{m}}^{b}M$ in the case $b \notin \mathbb{Z}$. We prove that $\mathcal{D}_{\mathbf{m}}^{b}M$ is irreducible if and only if $b + \mu \notin \mathbb{Z}$, see Theorem 3.6. Note that the action of $e_{21}(\mathbf{m})$ on $\mathcal{D}_{\mathbf{m}}M/M$ (resp. $\mathcal{D}_{\mathbf{m}}^{b}M$) is locally nilpotent (resp. bijective). We remark that all the modules constructed in [24] for different nonzero $\mu \in \mathbb{C}$ are actually all equivalent, giving essentially one module. The simple modules $\mathcal{D}_{\mathbf{m}}M/M$ and $\mathcal{D}_{\mathbf{m}}^{b}M$ in the present paper are very different from the module M constructed in [24], since the action of $e_{21}(\mathbf{m})$ on M is injective but not bijective.

We denote by $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{N}$ and $\mathbb{C}$ the sets of all integers, nonnegative integers, positive integers and complex numbers, respectively. For any Lie algebra L, we denote its universal enveloping algebra by $U(L)$.

2. Preliminaries

Let q be a non-zero complex number. Let $\mathbb{C}_q$ be the associative algebra generated by $t_1^{\pm 1}$, $t_2^{\pm 1}$ subject to the relation

$$t_1t_1^{-1} = t_1^{-1}t_1 = t_2t_2^{-1} = t_2^{-1}t_2 = 1,\ t_2t_1 = qt_1t_2.$$

For any $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2$, denote $\mathbf{t}^{\mathbf{m}} = t_1^{m_1}t_2^{m_2}$. Then it is clear that

$$\mathbf{t}^{\mathbf{m}}\mathbf{t}^{\mathbf{n}} = q^{m_2n_1}\mathbf{t}^{\mathbf{m}+\mathbf{n}} = q^{m_2n_1-m_1n_2}t^{\mathbf{n}}t^{\mathbf{m}}$$

for any $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^2$.

Let $\mathfrak{gl}_2(\mathbb{C}_q) := \mathfrak{gl}_2(\mathbb{C})\otimes_{\mathbb{C}}\mathbb{C}_q$ be the general linear Lie algebra coordinated by $\mathbb{C}_q$. Denote $X(\mathbf{m}) = X\otimes t^{\mathbf{m}}$ for $X \in \mathfrak{gl}_2(\mathbb{C})$, $\mathbf{m} \in \mathbb{Z}^2$. We identify X with $X(0)$. Then the Lie bracket of $\mathfrak{gl}_2(\mathbb{C}_q)$ is given by

$$[e_{ij}(\mathbf{m}), e_{kl}(\mathbf{n})] = \delta_{jk}q^{m_2n_1}e_{il}(\mathbf{m}+\mathbf{n}) - \delta_{il}q^{n_2m_1}e_{kj}(\mathbf{m}+\mathbf{n})$$

for $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^2$, $1 \le i, j, k, l \le 2$, where e_{ij} is the matrix whose (i, j)-entry is 1 and 0 elsewhere.

Clearly $\mathfrak{gl}_2(\mathbb{C}_q)$ is $\mathbb{Z}^2$-graded and to reflect this fact we add degree derivations. Let $\widehat{\mathfrak{gl}_2(\mathbb{C}_q)} = \mathfrak{gl}_2(\mathbb{C}_q) \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2$ and extend the Lie bracket as

$$[d_1, X(\mathbf{m})] = m_1X(\mathbf{m}),\quad [d_2, X(\mathbf{m})] = m_2X(\mathbf{m}).$$

The Lie subalgebra $[\mathfrak{gl}_2(\mathbb{C}_q), \mathfrak{gl}_2(\mathbb{C}_q)] \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2$ of $\mathfrak{g}$ is called an extended affine Lie algebra of type A_1 with nullity 2. (See [1,2] for definitions).

Let $\mathfrak{h} = \mathbb{C}e_{11} \oplus \mathbb{C}e_{22} \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2$ which is a Cartan subalgebra of $\mathfrak{g} := \widehat{\mathfrak{gl}_2(\mathbb{C}_q)}$. A $\mathfrak{g}$-module M is called a weight module if $\mathfrak{h}$ acts diagonally on M, i.e. $M = \oplus_{\lambda\in\mathfrak{h}^*}M_\lambda$, where $M_\lambda := \{v \in M \mid xv = \lambda(x)v, \forall\, x \in \mathfrak{h}\}$. A nonzero element $v \in M_\lambda$ is called a weight vector.

Let us denote

$$\mathfrak{n}_+ = e_{12} \otimes \mathbb{C}_q,\quad \mathfrak{n}_- = e_{21} \otimes \mathbb{C}_q,$$
$$\mathcal{H} = (e_{11} \otimes \mathbb{C}_q) \oplus (e_{22} \otimes \mathbb{C}_q) \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2.$$

Then

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathcal{H} \oplus \mathfrak{n}_-.$$

For a weight $\mathfrak{g}$-module M, a weight vector v in M is called a highest weight vector if $\mathfrak{n}_+v = 0$. The module M is called a highest weight module if it is generated by a highest weight vector.

Next, we will recall a class of highest weight modules over $\widehat{\mathfrak{gl}_2(\mathbb{C}_q)}$ defined by free fields, see [24].

Let

$$\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_{\mathbf{m}}, \mathbf{m} \in \mathbb{Z}^2]$$

be a polynomial ring with infinitely many variables $x_{\mathbf{m}}$. Denote by $\mathcal{A}(\mathbf{x})$ the associative algebra of formal power series of differential operators on $\mathbb{C}[\mathbf{x}]$. By [24], there is an algebra homomorphism

$$\Phi : U(\mathfrak{g}) \to \mathcal{A}(\mathbf{x})$$