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Branes through finite group actions

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ABSTRACT

Mid-dimensional (A, B, A) and (B, B, B)-branes in the moduli space of flat $G_{\mathbb{C}}$-connections appearing from finite group actions on compact Riemann surfaces are studied. The geometry and topology of these spaces are then described via the corresponding Higgs bundles and Hitchin fibrations.

1. Introduction — mid-dimensional (B, B, B) — *branes*

This paper is dedicated to the study of mid-dimensional subspaces of the neutral connected component of the moduli space of flat $G_{\mathbb{C}}$-connections on a compact Riemann surface Σ of genus $g \geq 2$, for $G_{\mathbb{C}}$ a complex Lie group, associated to finite group actions on Σ. As shown in [1,2], considering suitable stability conditions, a Higgs bundle defines a solution of equations for a G-connection A known as the Hitchin equations, where G is the maximal compact subgroup of $G_{\mathbb{C}}$. In particular, for $G = U(n)$ these are $F_A + [\Phi, \Phi^*] = 0$ and the vanishing of the antiholomorphic part of the covariant derivative of Φ, this is, $d''_A\Phi = 0$. In such case, the connection $\nabla_A + \Phi + \Phi^*$ is flat, with holonomy in $\mathrm{GL}(n, \mathbb{C})$. We shall denote by $\mathcal{M}_{G_{\mathbb{C}}}$ the moduli space of $G_{\mathbb{C}}$-Higgs bundles on a compact Riemann surface Σ, the space of solutions to the Hitchin equations on the surface modulo gauge equivalence.

The space of solutions to Hitchin's equations is a hyper-Kähler manifold, and thus there is a family of complex structures from which we shall fix I, J, K obeying quaternionic equations; along the paper we shall fix those structures following the notation of [1,3,4]. With this convention, the smooth locus of $\mathcal{M}_{G_{\mathbb{C}}}$ corresponds to the space of solutions to Hitchin's equations together with the complex structure I. Throughout this work we shall adopt the physicists' language in which a Lagrangian submanifold supporting a flat connection is called an *A-brane*, and a complex submanifold supporting a complex sheaf is a *B-brane*. By considering the support of branes, one may say that a submanifold of a hyper-Kähler manifold is of type A or B with respect to each of the structures, and hence one may speak of branes of type (B, B, B), (B, A, A), (A, B, A) and (A, A, B). Since understanding the support of branes is already a difficult endeavour, throughout the paper we shall consider only the support of branes and study their appearance within the moduli space of Higgs bundles. With a mild abuse of notation, we shall refer to the support of branes as branes themselves.

The construction and study of branes in the moduli space of $G_{\mathbb{C}}$-Higgs bundles through actions on the group $G_{\mathbb{C}}$ or on the surface Σ only began a few years ago — see, for example, [5] for a first appearance of branes through actions on the

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surface, and [3] for actions on groups. Whilst one may describe those branes inside the Hitchin fibration of $\mathcal{M}_{G_\mathbb{C}}$ (obtaining for instance a nonabelian fibration of (B, A, A) as described in [6], or a real integrable system through (A, B, A)-branes as in [5]) not much more is known about the geometry of those branes constructed, or how other interesting subspaces appear through group actions.

The construction of hyperholomorphic branes, or (B, B, B)-branes, is of particular interest, as not many nontrivial examples have been found — one may construct (B, B, B)-branes by considering the moduli space of $H_\mathbb{C}$-Higgs bundles inside the moduli space of $G_\mathbb{C}$-Higgs bundles, for $H_\mathbb{C}$ a complex subgroup of $G_\mathbb{C}$, but how else may one define (B, B, B)-branes? After overviewing finite group actions on flat connections, in Section 2.2 new (B, B, B)-branes are constructed:

Theorem 8. *Let Σ be a compact Riemann surface of genus $g \geq 2$ and Γ a finite group acting on Σ by holomorphic automorphisms. The connected components of the space of gauge equivalence classes of irreducible Γ-equivariant flat $G_\mathbb{C}$-connections are hyper-Kähler submanifolds of the moduli space of flat irreducible $G_\mathbb{C}$-connections on Σ, and hence give (B, B, B)-branes in the moduli space of $G_\mathbb{C}$-Higgs bundles.*

Since mid-dimensional spaces may be A-branes or B-branes with respect to each of the structures, it is particularly interesting to seek finite group actions giving mid-dimensional hyper-Kähler submanifolds. In Section 3 we give a classification of actions leading to mid-dimensional branes in the moduli space $\mathcal{M}^g_{dR}$ of flat $\mathrm{SL}(2, \mathbb{C})$-connections on a compact Riemann surface of genus g.

Theorem 11. *Let Σ be a compact Riemann surface of genus $g \geq 2$ and Γ be a finite group acting on Σ by holomorphic automorphisms such that a component of the moduli space of Γ-equivariant flat $\mathrm{SL}(2, \mathbb{C})$-connections on Σ has half the dimension of the moduli space $\mathcal{M}^g_{dR}$. Then one of the following holds:*

(I) *$\Gamma = \mathbb{Z}_2$ acts by a fixed points free involution on Σ, or*
(II) *Σ is hyperelliptic of genus 3 and $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$.*

In case (II), one of the $\mathbb{Z}_2$-factors corresponds to the hyperelliptic involution, whilst the other $\mathbb{Z}_2$-factor corresponds to an involution with 4 fixed points.

Although we highlight results which hold for generic groups, most of the work in the remaining sections is done for $G_\mathbb{C} = \mathrm{SL}(2, \mathbb{C})$. In order to understand the geometry of these branes, we consider them inside the moduli space of Higgs bundles and look at their intersection with smooth fibres of the Hitchin fibration. After overviewing the $\mathrm{SL}(2, \mathbb{C})$-Hitchin fibration in Section 4, in Section 5 we obtain the following geometric description of the intersection of the (B, B, B)-brane of Theorem 11(I) with the regular fibres of the Hitchin fibration:

Theorem 14. *Let τ be a fixed point free involution. Then, the τ-equivariant (B, B, B)-brane intersects a generic fibre of the Hitchin fibration over a point defining the spectral curve S in the abelian variety* $\mathrm{Prym}(S/\tau, \Sigma/\tau)/\mathbb{Z}_2$.

In order to study the equivariant (B, B, B)-branes of Theorem 11(II), it is shown in Section 6 that it suffices to consider Higgs bundles over hyperelliptic surfaces of genus 3 with fixed point free actions. In such case, we describe the intersection of the brane with the regular fibres of the Hitchin fibration in Theorem 25. Finally, from [4, Section 12], the equivariant and anti-equivariant spaces considered in this short paper have dual branes in the moduli spaces of Higgs bundles for the Langlands dual group.

We conclude the work in Section 7 with comments on Langlands duality for the branes constructed in this paper, and noting it is interesting to compare the spaces in Theorem 8 with spaces appearing in other papers - e.g. the real integrable systems given by (A, B, A)-branes in [5, Theorem 17] and the (B, A, A)-branes of [3].

Since the work of the present paper was first announced at the Simons Center of Geometry and Physics in June 2016 (at the conference "New perspectives on Higgs bundles, branes and quantization"), actions of finite groups on the moduli space of Higgs bundles and of flat connections have received increasing attention — an interested reader in the subject might want to look among other papers at the work of Schaffhauser [7] for actions on moduli spaces of vector bundles, and of Hoskins–Schaffhauser [8] for interesting branes arising from group actions on quiver varieties.

2. Equivariant flat connections

Consider Σ a compact (connected) Riemann surface of genus $g \geq 2$ and a finite group action $\Gamma \times \Sigma \to \Sigma$. These actions have been studied by many researchers and in the case of surfaces of genus 2 and 3, a complete classification of all finite group actions is given in [9, Tables 4, 5]. Moreover, in the case of actions induced on rank 2 bundles through automorphisms of Σ, a very concrete description of the fixed points in terms of parabolic structures is given in [10].

In order to understand flat equivariant $G_\mathbb{C}$-connections on Σ, one needs to first fix a C^∞ trivialization $\underline{\mathbb{C}}^n = \Sigma \times \mathbb{C}^n \to \Sigma$ of the underlying vector bundle. In what follows we shall restrict our attention to the groups $G_\mathbb{C} = \mathrm{GL}(n, \mathbb{C})$, $\mathrm{SL}(n, \mathbb{C})$, and thus in the case of $\mathrm{SL}(n, \mathbb{C})$ require the trivialization to preserve the determinant.