Journal of Geometry and Physics 129 (2018) 192–199

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys

# Weak completions, bornologies and rigid cohomology

Guillermo Cortiñas [a], Joachim Cuntz [b,*], Ralf Meyer [c], Georg Tamme [d]

[a] *Dep. Matemática-IMAS, FCEyN-UBA, Ciudad Universitaria Pab 1, 1428 Buenos Aires, Argentina*
[b] *Mathematisches Institut, Westfälische Wilhelms-Universität Münster, Einsteinstr. 62, 48149 Münster, Germany*
[c] *Mathematisches Institut, Georg-August Universität Göttingen, Bunsenstraße 3–5, 37073 Göttingen, Germany*
[d] *Universität Regensburg, Fakultät für Mathematik, 93040 Regensburg, Germany*

ARTICLE INFO

*Article history:*
Received 12 January 2018
Accepted 6 March 2018
Available online 20 March 2018

Dedicated to Alain Connes on the occasion of his 70th birthday

*MSC:*
14F30
14F40
13D03

*Keywords:*
Algebraic geometry
Positive characteristic
Rigid cohomology
Overconvergent completions
Bornological algebras
Cyclic homology

ABSTRACT

Let $V$ be a complete discrete valuation ring with residue field $k$ of positive characteristic and with fraction field $K$ of characteristic 0. We clarify the analysis behind the Monsky–Washnitzer completion of a commutative $V$-algebra using completions of bornological $V$-algebras. This leads us to a functorial chain complex for a finitely generated commutative algebra over the residue field $k$ that computes its rigid cohomology in the sense of Berthelot.


## 1. Introduction

The problem of defining a cohomology theory ($p$-adic admitting a Frobenius operator) with good properties for an algebraic variety over a field $k$ of non-zero characteristic $p$ has a long history. In the breakthrough paper [1] by Monsky and Washnitzer, such a theory for smooth affine varieties was constructed as follows. Take a complete discrete valuation ring $V$ of mixed characteristic with uniformizer $\pi$ and residue field $k = V/\pi V$ (for example, $V$ the ring of Witt vectors $W(k)$ if $k$ is perfect). Let $K$ be the fraction field of $V$. Choose a $V$-algebra $R$ which is a lift mod $\pi$ of the coordinate ring of the variety and which is smooth over $V$ (such a lift exists by [2]). Monsky–Washnitzer then introduce the 'weak' or dagger-completion $R^\dagger$ of $R$ and define their cohomology as the de Rham cohomology of $R^\dagger \otimes_V K$. The construction of a weak completion has become a basis for the definition of cohomology theories in this context ever since. The Monsky–Washnitzer theory has been generalized by Berthelot [3] to "rigid cohomology", which is a satisfactory cohomology theory for general varieties over $k$. Its definition uses certain de Rham complexes on rigid analytic spaces.

* Corresponding author.
*E-mail addresses:* gcorti@dm.uba.ar (G. Cortiñas), cuntz@math.uni-muenster.de (J. Cuntz), rmeyer2@uni-goettingen.de (R. Meyer), georg.tamme@ur.de (G. Tamme).
*URL:* http://mate.dm.uba.ar/~gcorti (G. Cortiñas).

https://doi.org/10.1016/j.geomphys.2018.03.005
0393-0440/

The definition of the Monsky–Washnitzer cohomology as well as Berthelot's definition of rigid cohomology depend on choices. The chain complexes that compute them are not functorial for algebra homomorphisms. Only their homology is functorial. Functorial complexes that compute rigid cohomology have been constructed by Besser [4]. However, the construction is based on some abstract existence statements, and is not at all explicit.

In [5] we had developed a general framework for bornological structures on $V$-algebras which allows in particular to generalize the weak completions of Monsky–Washnitzer to bornological versions of $J$-adic completions for an ideal $J$ in a $V$-algebra. We had used this to study cyclic homology for such completions and to relate rigid cohomology to cyclic homology in this setting. As one major result we also had constructed a natural and explicit chain complex computing rigid cohomology for affine varieties over $k$. However [5] contained much more material than was needed to this latter end and the route to the construction of this complex was not the most direct possible. In the last section of [5] we had sketched a more direct approach. It is the aim of the present article to make this sketch explicit. At the same time we show that some of the basic arguments in [5] can be simplified significantly if one restricts the technical analysis to what is needed for that purpose. We obtain a conceptually simple construction of the natural complex computing rigid cohomology which is very much in the spirit of Grothendieck's infinitesimal cohomology for non-smooth varieties.

We proceed as follows. For a commutative $V$-algebra $R$ we write $\underline{R}$ for $R \otimes K$. Given a $k$-algebra $A$, we consider a presentation $J \to P \twoheadrightarrow A$ by a free commutative $V$-algebra $P$. For the $m$th powers of the kernel $J$, we obtain a projective system $(\overline{\underline{P}_{J^m}})$ of bornological $J^m$-adic completions. Since the maps in this system are not surjective, the natural description of the de Rham cohomology of this pro-algebra is not via the de Rham complex of the projective limit but rather via the homotopy projective limit of the de Rham complexes (which is given as an explicit chain complex). As a consequence of homotopy invariance, this homotopy limit does not depend on the choice of a free presentation $P$ up to quasi-isomorphism. A functorial choice is $P = V[A]$. One advantage of our approach is the fact that the formalism using bornological $J$-adic completions works fine also for algebras that are not Noetherian, such as $V[A]$.

For the special choice of free presentation given by $V[A]$, the homotopy limit of de Rham complexes mentioned above is completely explicit and manifestly functorial in $A$. Using results by Große-Klönne in [6] we show that it reproduces Berthelot's rigid cohomology.

### 1.1. Notation

Let $V$ be a complete discrete valuation ring and let $\pi$ be a generator for the maximal ideal in $V$. Let $K$ be the fraction field of $V$, that is, $K = V[\pi^{-1}]$. Let $k = V/\pi V$ be the residue field. We will always assume that $K$ has characteristic 0. Every element of $K$ is written uniquely as $x = u\pi^{\nu(x)}$, where $u \in V \setminus \pi V$ and $\nu(x) \in \mathbb{Z} \cup \{\infty\}$ is the *valuation* of $x$. We fix $0 < \epsilon < 1$, and define the *absolute value* $|\;|: K \to \mathbb{R}_{\geq 0}$ by $|x| = \epsilon^{\nu(x)}$ for $x \neq 0$ and $|0| = 0$.

If $M$ is a $V$-module, let $\underline{M}$ be the associated $K$-vector space $M \otimes K$. A $V$-module $M$ is *flat* if and only if the canonical map $M \to \underline{M}$ is injective, if and only if it is *torsion-free*, that is, $\pi x = 0$ implies $x = 0$.

Even though much of the foundational discussion works in greater generality, we will always assume that all algebras are commutative.

## 2. Bornological modules over discrete valuation rings

**Definition 2.1.** Let $M$ be a $V$-module. A (convex) *bornology* on $M$ is a family $\mathcal{B}$ of subsets of $M$, called *bounded* subsets, satisfying

- every finite subset is in $\mathcal{B}$;
- subsets and finite unions of sets in $\mathcal{B}$ are in $\mathcal{B}$;
- if $S \in \mathcal{B}$, then the $V$-submodule generated by $S$ also belongs to $\mathcal{B}$.

A *bornological* $V$*-module* is a $V$-module equipped with a bornology.

A $V$-linear map between bornological modules is said to be bounded if it maps bounded sets to bounded sets. A *bornological* $K$*-vector space* is a bornological $V$-module such that multiplication by $\pi$ is an invertible map with bounded inverse.

A bornological module equipped with an algebra structure is a bornological algebra, if the product of any two sets in $\mathcal{B}$ is again in $\mathcal{B}$ (i.e. if multiplication is bounded).

**Definition 2.2.** Let $X$ be a bornological $V$-module. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $X$ and let $x \in X$. If $S \subseteq X$ is bounded, then $(x_n)_{n\in\mathbb{N}}$ $S$*-converges* to $x$ if there is a sequence $(\delta_n)_{n\in\mathbb{N}}$ in $V$ with $\lim \delta_n = 0$ in the $\pi$-adic topology and $x_n - x \in \delta_n \cdot S$ for all $n \in \mathbb{N}$. A sequence in $X$ *converges* if it $S$-converges for some bounded subset $S$ of $X$. If $S$ is bounded, then $(x_n)_{n\in\mathbb{N}}$ is $S$*-Cauchy* if there is a sequence $(\delta_n)_{n\in\mathbb{N}}$ in $V$ with $\lim \delta_n = 0$ and $x_n - x_m \in \delta_l \cdot S$ for all $n, m, l \in \mathbb{N}$ with $n, m \geq l$. A sequence in $X$ is *Cauchy* if it is $S$-Cauchy for some bounded $S \subseteq X$. The bornological $V$-module $X$ is *separated* if limits of convergent sequences are unique. It is *complete* if it is separated and for every bounded $S \subseteq X$ there is a bounded $S' \subseteq X$ so that all $S$-Cauchy sequences are $S'$-convergent.

**Definition 2.3.** The *completion* of a bornological $V$-module $X$ is a complete bornological $V$-module $\overline{X}$ with a bounded map $X \to \overline{X}$ that is universal in the sense that any map from $X$ to a complete bornological $V$-module factors uniquely through it.