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On split regular BiHom-Lie superalgebras

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ABSTRACT

We introduce the class of split regular BiHom-Lie superalgebras as the natural extension of the one of split Hom-Lie superalgebras and the one of split Lie superalgebras. By developing techniques of connections of roots for this kind of algebras, we show that such a split regular BiHom-Lie superalgebra L is of the form $L = U + \sum_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}$ with U a subspace of the Abelian (graded) subalgebra H and any $I_{[\alpha]}$, a well described (graded) ideal of L, satisfying $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$. Under certain conditions, in the case of L being of maximal length, the simplicity of the algebra is characterized and it is shown that L is the direct sum of the family of its simple (graded) ideals.

1. Introduction

Hom-algebras appeared in the study of q-deformations of algebras of vector fields, in particular q-deformations of Witt and Virasoro algebras, see for instance [1–3]. So far, many authors have studied Hom-type algebras motivated in part for their applications in physics [4–11]. A BiHom-algebra is an algebra in such a way that the identities defining the structure are twisted by two homomorphisms ϕ, ψ. The notion of BiHom-Lie algebras was introduced in [12], which is intimately related to both Lie algebras and Hom-Lie algebras. The case of $\phi = \psi = \mathrm{Id}$ implies BiHom-Lie algebras are Lie algebras and the other case of $\phi = \psi$ give Hom-Lie algebras.

As is well-known, the class of the split algebras is specially related to addition quantum numbers, graded contractions and deformations. Determining the structure of split algebras will become more and more meaningful in the area of research in mathematical physics. Recently, the structure of different classes of split algebras have been studied by using techniques of connections of roots (see for instance [13–21]). The purpose of this paper is to consider the structure of split regular BiHom-Lie superalgebras by the techniques of connections of roots based on some work in [14–16].

Throughout this paper, split regular BiHom-Lie superalgebras L are considered of arbitrary dimension and over an arbitrary base field $\mathbb{K}$. This paper is organized as follows. In Section 2, we establish the preliminaries on split regular BiHom-Lie superalgebras theory. In Section 3, we show that such an arbitrary split regular BiHom-Lie superalgebra L with a symmetric root system is of the form $L = U + \sum_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}$ with U a subspace of the Abelian (graded) subalgebra H and any $I_{[\alpha]}$ a well described (graded) ideal of L, satisfying $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$. In Section 4, we show that under certain

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conditions, in the case of L being of maximal length, the simplicity of the algebra is characterized and it is shown that L is the direct sum of the family of its simple (graded) ideals.

2. Preliminaries

First we recall the definition of BiHom-Lie algebras and give the definition of BiHom-Lie superalgebras.

Definition 2.1 ([12]). A **BiHom-Lie algebra** over a field $\mathbb{K}$ is a 4-tuple $(L, [\cdot,\cdot], \phi, \psi)$, where L is a $\mathbb{K}$-linear space, $[\cdot,\cdot] : L \times L \to L$ a bilinear map and $\phi, \psi : L \to L$ linear mappings satisfying the following identities:

1. $\phi \circ \psi = \psi \circ \phi$,
2. $[\psi(x), \phi(y)] = -[\psi(y), \phi(x)]$, (BiHom-skew-symmetry)
3. $[\psi^2(x), [\psi(y), \phi(z)]] + [\psi^2(y), [\psi(z), \phi(x)]] + [\psi^2(z), [\psi(x), \phi(y)]] = 0$, (BiHom-Jacobi identity)

for any $x, y, z \in L$.

Definition 2.2. A **BiHom-Lie superalgebra** L is a $\mathbb{Z}_2$-graded algebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ endowed with an even bilinear mapping $[\cdot,\cdot] : L \times L \to L$ and two homomorphisms $\phi, \psi : L \to L$ such that

1. $[x, y] \subset L_{\bar{i}+\bar{j}}$,
2. $\phi \circ \psi = \psi \circ \phi$,
3. $[\psi(x), \phi(y)] = -(-1)^{\bar{i}\bar{j}}[\psi(y), \phi(x)]$, (BiHom-super-skew-symmetry)
4. $(-1)^{\bar{k}\bar{i}}[\psi^2(x), [\psi(y), \phi(z)]] + (-1)^{\bar{i}\bar{j}}[\psi^2(y), [\psi(z), \phi(x)]] + (-1)^{\bar{j}\bar{k}}[\psi^2(z), [\psi(x), \phi(y)]] = 0$, (BiHom-super-Jacobi identity)

for any homogenous element $x \in L_{\bar{i}}, y \in L_{\bar{j}}, z \in L_{\bar{k}}, \bar{i}, \bar{j}, \bar{k} \in \mathbb{Z}_2$. When ϕ, ψ furthermore are algebra automorphisms it is said that L is a **regular BiHom-Lie superalgebra**.

Lie superalgebras are examples of BiHom-Lie superalgebras by taking $\phi = \psi = \mathrm{Id}$. Hom-Lie superalgebras are also examples of BiHom-Lie superalgebras by considering $\psi = \phi$.

Example 2.3. Let $(L, [\cdot,\cdot])$ be a Lie superalgebra, $\phi, \psi : L \to L$ two automorphisms and $\phi \circ \psi = \psi \circ \phi$. If we endow the underlying linear space L with a new product $[\cdot,\cdot]' : L \times L \to L$ defined by $[x, y]' := [\phi(x), \psi(y)]$ for any $x, y \in L$, we have that $(L, [\cdot,\cdot]', \phi, \psi)$ becomes a regular BiHom-Lie superalgebra.

Throughout this paper we will consider a regular BiHom-Lie superalgebra L being of arbitrary dimension and over an arbitrary base field $\mathbb{K}$. $\mathbb{N}$ denotes the set of all non-negative integers and $\mathbb{Z}$ denotes the set of all integers.

Note that $L_{\bar{0}}$ is a BiHom-Lie algebra called the even or bosonic part of L, while $L_{\bar{1}}$ is called the odd or fermonic part of L. The usual regularity concepts will be understood in the graded sense. That is, a subalgebra A of L is a graded subspace $A = A_{\bar{0}} \oplus A_{\bar{1}}$ of L, such that $[A, A] \subset A$ and $\phi(A) = \psi(A) = A$. A graded subspace $I = I_{\bar{0}} \oplus I_{\bar{1}}$ of L is called an ideal if $[I, L] + [L, I] \subset I$ and $\phi(I) = \psi(I) = I$. A BiHom-Lie superalgebra L will be called simple if $[L, L] \neq 0$ and its only (graded) ideals are $\{0\}$ and L.

Let us introduce the class of split algebras in the framework of regular BiHom-Lie superalgebras L. First, we recall that a Lie superalgebra $(L, [\cdot,\cdot])$, over a base field $\mathbb{K}$, is called split respect to a maximal Abelian (graded) subalgebra $H = H_{\bar{0}} \oplus H_{\bar{1}}$ of L, if L can be written as the direct sum

$$L = H \oplus (\oplus_{\alpha \in \Gamma} L_\alpha)$$

where

$$L_\alpha = \{v_\alpha \in L : [h_{\bar{0}}, v_\alpha] = \alpha(h_{\bar{0}})v_\alpha \textit{ for any } h_{\bar{0}} \in H_{\bar{0}}\}.$$

being any $\alpha : H_{\bar{0}} \to \mathbb{K}, \alpha \in \Gamma$, a nonzero linear functional on $H_{\bar{0}}$ such that $L_\alpha \neq 0$.

We introduce the concept of a split regular BiHom-Lie superalgebra in an analogous way.

Definition 2.4. Denote by $H = H_{\bar{0}} \oplus H_{\bar{1}}$ a maximal Abelian (graded) subalgebra, of a regular BiHom-Lie superalgebra L. For a linear functional $\alpha : H_{\bar{0}} \to \mathbb{K}$, we define the root space of L (with respect to H) associated to α as the subspace

$$L_\alpha = \{v_\alpha \in L : [h_{\bar{0}}, \phi(v_\alpha)] = \alpha(h_{\bar{0}})\phi\psi(v_\alpha) \textit{ for any } h_{\bar{0}} \in H_{\bar{0}}\}.$$

The elements $\alpha : H_{\bar{0}} \to \mathbb{K}$ satisfying $L_\alpha \neq 0$ are called roots of L with respect to H. We denote $\Lambda := \{\alpha \in (H_{\bar{0}})^* \setminus \{0\} : L_\alpha \neq 0\}$. We say that L is a **split regular BiHom-Lie superalgebra**, with respect to H, if

$$L = H \oplus (\oplus_{\alpha \in \Lambda} L_\alpha).$$

We also say that Λ is the root system of L.

Note that when $\phi = \psi = \mathrm{Id}$, the split Lie superalgebras become examples of split regular BiHom-Lie superalgebras and when $\phi = \psi$, the split regular Hom-Lie superalgebras become examples of split regular BiHom-Lie superalgebras. Split regular BiHom-Lie algebras are examples of split regular BiHom-Lie superalgebras. Hence, the present paper extends the results in [15,16]. Let us see another example.