Journal of Geometry and Physics 128 (2018) 48–57

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys

On the asymptotically Poincaré-Einstein 4-manifolds with harmonic curvature

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HIGHLIGHTS

- We investigate the AH mass of an APE 4-manifold with harmonic curvature.
- We prove a rigidity of complete non-compact 4-manifold with harmonic curvature and non-positive scalar curvature.
- We prove a rigidity of complete non-compact Einstein 4-manifold with non-positive scalar curvature.
- We prove a rigidity of an APE 4-manifold with harmonic curvature. The bound of the L^2-norm of the curvature is a highlight.

ARTICLE INFO

Article history:
Received 10 November 2017
Accepted 10 February 2018
Available online 16 February 2018

MSC:
primary 53C25
53C24

Keywords:
Asymptotically Poincaré-Einstein manifold
Harmonic curvature
Mass aspect tensor
Rigidity

ABSTRACT

In this paper, we discuss the mass aspect tensor and the rigidity of an asymptotically Poincaré-Einstein (APE) 4-manifold with harmonic curvature. We prove that the trace-free part of the mass aspect tensor of an APE 4-manifold with harmonic curvature and normalized Einstein conformal infinity is zero. As to the rigidity, we first show that a complete noncompact Riemannian 4-manifold with harmonic curvature and positive Yamabe constant as well as a L^2-pinching condition is Einstein. As an application, we then obtain that an APE 4-manifold with harmonic curvature and positive Yamabe constant is isometric to the hyperbolic space provided that the L^2-norm of the traceless Ricci tensor or the Weyl tensor is small enough and the conformal infinity is a standard round 3-sphere.

1. Introduction

Poincaré-Einstein (PE) manifolds have been deeply investigated recently because of the so called AdS/CFT correspondence proposed in the theory of quantum gravity in theoretical physics. Since an asymptotically Poincaré-Einstein (APE) manifold and a Riemannian manifold with harmonic curvature are both a kind of generalization of a PE manifold, we wonder whether an APE manifold with harmonic curvature is Einstein. In this paper, we mainly answer this question and investigate the mass aspect tensor and the rigidity of an APE 4-manifold with harmonic curvature.

Before considering the APE Riemannian manifold, we first give some basic materials about conformally compact manifold. Suppose that M^n can be realized as the interior of a smooth compact manifold $\bar{M}^n$ with boundary $\partial\bar{M}$. A defining function τ for the boundary $\partial\bar{M}$ is a smooth function on $\bar{M}^n$ such that $\tau > 0$ in M^n; $\tau = 0$ on $\partial\bar{M}$; $d\tau \neq 0$ on $\partial\bar{M}$. We refer to $\partial\bar{M}$ as the boundary-at-infinity of M^n and denote it by $\partial_\infty M$.

A complete noncompact Riemannian metric g on M^n is said to be $C^{k,\mu}$ conformally compact if the compactified metric $\bar{g} = \tau^2 g$ extends to be a $C^{k,\mu}$ Riemannian metric on $\bar{M}^n$. If in addition, $|d\tau|^2_{\tau^2 g} = 1$ on $\partial_\infty M$, then (M^n, g) is called $C^{k,\mu}$ asymptotically hyperbolic (AH).

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https://doi.org/10.1016/j.geomphys.2018.02.008
0393-0440/

The metric $\bar{g} = \tau^2 g$ induces a metric $\hat{g}$ on the boundary $\partial_\infty M$, and the metric g induces a conformal class of metric $[\hat{g}]$ on the boundary $\partial_\infty M$ when defining functions vary. The conformal boundary manifold $\left(\partial_\infty M, [\hat{g}]\right)$ is called the conformal infinity of the conformally compact manifold (M^n, g).

Given a C^1 AH metric g and a representative $\hat{g}$ in $[\hat{g}]$ on the conformal boundary $\partial_\infty M$, there is a uniquely determined defining function r such that $|dr|^2_{r^2 g} \equiv 1$ in a neighborhood of the boundary. With this choice of r, in a collar neighborhood $[0, \epsilon) \times \partial_\infty M \subset M^n$ for some $\epsilon > 0$, g takes the geodesic normal form

$$g = r^{-2}\left(dr^2 + g_r\right), \tag{1}$$

where g_r is a curve of metrics on $\partial_\infty M$ with $g_r|_{r=0} = \hat{g}$. Such defining function r is called the geodesic defining function associated with $\hat{g}$.

Since the sectional curvatures of an AH metric approach -1 as $r \to 0$, the Ricci curvature will approach $-(n-1)g$ at infinity. Following [1,2], we now turn our attention to the AH manifolds that have Ricci curvature sufficiently pinched near infinity.

Definition 1.1. An AH metric g on M^n is called Poincaré-Einstein (PE) if $E(g) := Ric(g) + (n-1)g$ vanishes identically. It is called asymptotically Poincaré-Einstein (APE) if $|E|_g = O(r^n)$.

Let (M^n, g) be an APE manifold and $g = r^{-2}\left(dr^2 + g_r\right)$ with $g_r|_{r=0} = \hat{g}$. Assume that g_r is sufficiently regular that its asymptotical expansion may be calculated by the APE condition. Let E_{00} denote $E\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right)$ and $E^\perp$ denote the projection of E orthogonal to $\frac{\partial}{\partial r}$. A straightforward calculation shows that

$$E_{00} = -\frac{1}{2} tr_{g_r} g_r'' + \frac{1}{2r} tr_{g_r} g_r' + \frac{1}{4}|g_r'|^2_{g_r} \tag{2}$$

and

$$E^\perp = Ric(g_r) - \frac{1}{2}g_r'' + \frac{n-2}{2r}g_r' + \frac{1}{2r}g_r tr_{g_r} g_r' + \frac{1}{2}g_r' \cdot g_r^{-1} \cdot g_r' - \frac{1}{4}g_r' tr_{g_r} g_r', \tag{3}$$

where $'$ denotes $\frac{\partial}{\partial r}$ and $g_r' \cdot g_r^{-1} \cdot g_r'$ denotes the tensor with components $(g_r)'_{ik}(g_r)^{kl}(g_r)'_{lj}$.

By similar computation as that in [3], Bahuaud, Mazzeo and Woolgar showed in [1, Proposition 2.2] that for an APE metric g, the expansion of g_r has the following form:

$$g_r = \hat{g} + g^{(2)}r^2 + (\text{even powers of } r) + g^{(n-1)}r^{n-1} + O\left(r^n\right) \tag{4}$$

when n is even; and

$$g_r = \hat{g} + g^{(2)}r^2 + (\text{even powers of } r) + \left(g^{(n-1)} + \tilde{g}^{(n-1)}\log r\right) r^{n-1} + O\left(r^n \log r\right) \tag{5}$$

when n is odd, where

(1) $g^{(2k)}$ are determined by $\hat{g}$ for $2k < n-1$ and $-g^{(2)}$ is the Schouten tensor $P\left(\hat{g}\right)$ of $\hat{g}$ for $n \geq 3$, i.e.

$$g^{(2)} = -P\left(\hat{g}\right) = -\frac{1}{n-2}\left(Ric\left(\hat{g}\right) - \frac{R\left(\hat{g}\right)}{2(n-1)}\hat{g}\right);$$

(2) when n is even, $g^{(n-1)}$ is undetermined but $tr_{\hat{g}} g^{(n-1)} = 0$;
(3) when n is odd, the ambient obstruction tensor $\tilde{g}^{(n-1)}$ is determined by $\hat{g}$ and $tr_{\hat{g}}\tilde{g}^{(n-1)} = 0$;
(4) when n is odd, $tr_{\hat{g}} g^{(n-1)}$ is determined by the prior coefficients, but the trace-free part of $g^{(n-1)}$ is undetermined.

Notice that when n is odd, g_r will always have an expansion involving $\log r$. But as the ambient obstruction tensor $\tilde{g}^{(n-1)}$ is determined by $\hat{g}$, if we choose $\hat{g}$ carefully, for example, we choose $\hat{g}$ to be Einstein, then we will have $\tilde{g}^{(n-1)} = 0$. We consider in this paper a class of APE manifolds which have special conformal infinities, that is, the representative metric $\hat{g}$ is Einstein, normalized such that $Ric\left(\hat{g}\right) = \lambda(n-2)\hat{g}$ with $\lambda \in \{-1, 0, 1\}$. We call such $\left(\partial_\infty M, [\hat{g}]\right)$ the normalized Einstein conformal infinity.

We will also need the following notion of AH mass for an APE Riemannian manifold developed in [2].

Lemma 1.2 *([2, Lemma 2.2 and Definition 2.5]). Let (M^n, g) be an APE Riemannian manifold with normalized Einstein conformal infinity $\left(\partial_\infty M, [\hat{g}]\right)$, then g has the form*

$$g = r^{-2}\left(dr^2 + \left(1 - \frac{\lambda}{4}r^2\right)^2 \hat{g} + \frac{1}{n-1}r^{n-1}\theta + \frac{1}{n}r^n\kappa + O\left(r^{n+1}\right)\right) \tag{6}$$

where θ is the Neumann data for g with $tr_{\hat{g}}\theta = 0$ and κ is the mass aspect tensor for g. Moreover, $tr_{\hat{g}}\kappa$ is the mass aspect function and $\int_{\partial_\infty M} tr_{\hat{g}}\kappa\, dv_{\hat{g}}$ is the mass for g.