Journal of Geometry and Physics 127 (2018) 19–31

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys

Review

Characteristic distribution: An application to material bodies

Víctor Manuel Jiménez [a,*], Manuel de León [a], Marcelo Epstein [b]

[a] Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM), c\| Nicolás Cabrera, 13-15, Campus Cantoblanco, UAM, 28049 Madrid, Spain
[b] Department of Mechanical Engineering, University of Calgary, 2500 University Drive NW, Calgary, Alberta, Canada, T2N IN4

ARTICLE INFO

Article history:
Received 24 November 2017
Received in revised form 19 January 2018
Accepted 23 January 2018
Available online 3 February 2018

Keywords:
Smooth distribution
Singular foliation
Groupoid
Uniformity
Material groupoid

ABSTRACT

Associated to each material body $\mathcal{B}$ there exists a groupoid $\Omega\left(\mathcal{B}\right)$ consisting of all the material isomorphisms connecting the points of $\mathcal{B}$. The uniformity character of $\mathcal{B}$ is reflected in the properties of $\Omega\left(\mathcal{B}\right)$: $\mathcal{B}$ is uniform if, and only if, $\Omega\left(\mathcal{B}\right)$ is transitive. Smooth uniformity corresponds to a Lie groupoid and, specifically, to a Lie subgroupoid of the groupoid $\Pi^1\left(\mathcal{B},\mathcal{B}\right)$ of 1-jets of $\mathcal{B}$. We consider a general situation when $\Omega\left(\mathcal{B}\right)$ is only an algebraic subgroupoid. Even in this case, we can cover $\mathcal{B}$ by a material foliation whose leaves are transitive. The same happens with $\Omega\left(\mathcal{B}\right)$ and the corresponding leaves generate transitive Lie groupoids (roughly speaking, the leaves covering $\mathcal{B}$). This result opens the possibility to study the homogeneity of general material bodies using geometric instruments.

Contents

1. Introduction 19
2. Groupoids 20
3. Characteristic distribution 23
4. Material groupoid and material distribution 27
5. Examples 29
Acknowledgements 31
References 31

1. Introduction

As it is well-known, associated to any simple material body $\mathcal{B}$ there exists a groupoid $\Omega\left(\mathcal{B}\right)$ over $\mathcal{B}$ called the *material groupoid of* $\mathcal{B}$ (see for example [1–3] or [4]). A material body is *simple* (or of grade 1) if the mechanical response functional at each point depends on the deformation gradient alone (and not on higher gradients). $\Omega\left(\mathcal{B}\right)$ consists of all linear isomorphisms P between the tangent spaces $T_X\mathcal{B}$ and $T_Y\mathcal{B}$ such that

$$W\left(FP,X\right)=W\left(F,Y\right),$$

for any deformation gradient F at Y, where X, Y run along the body $\mathcal{B}$. Here, W is the mechanical response of the body $\mathcal{B}$, typically the stored energy per unit mass.

* Corresponding author.
E-mail addresses: victor.jimenez@icmat.es (V.M. Jiménez), mdeleon@icmat.es (M. de León), epstein@enme.ucalgary.ca (M. Epstein).

https://doi.org/10.1016/j.geomphys.2018.01.021
0393-0440/

The uniformity of $\mathcal{B}$ is reflected on the properties of the material groupoid $\Omega\left(\mathcal{B}\right)$. In particular, $\mathcal{B}$ is (smoothly) uniform if, and only if, $\Omega\left(\mathcal{B}\right)$ is a transitive (Lie) subgroupoid of $\Pi^1\left(\mathcal{B},\mathcal{B}\right)$, where $\Pi^1\left(\mathcal{B},\mathcal{B}\right)$ is the Lie groupoid over $\mathcal{B}$, called 1-*jets groupoid on* $\mathcal{B}$, of all linear isomorphisms P between the tangent spaces $T_X\mathcal{B}$ and $T_Y\mathcal{B}$, for $X, Y \in \mathcal{B}$.

In this paper, we consider a more general situation. We study the problem from a purely mathematical framework, since we are convinced that this analysis should be relevant not only for its applications to Continuum Mechanics, but also for the general theory of groupoids.

So, let $\overline{\Gamma} \subseteq \Gamma$ be a subgroupoid of a Lie groupoid $\Gamma \rightrightarrows M$; notice that we are not assuming, in principle, any differentiable structure on $\overline{\Gamma}$. Even in that case, we can construct a generalized distribution $A\overline{\Gamma}^T$ over Γ generated by the (local) left-invariant vector fields on Γ whose flow at the identities is totally contained in $\overline{\Gamma}$. This distribution $A\overline{\Gamma}^T$ will be called the *characteristic distribution of* $\overline{\Gamma}$. Due to the groupoid structure, we can still associate two new objects to $A\overline{\Gamma}^T$, denoted by $A\overline{\Gamma}$ and $A\overline{\Gamma}^\sharp$, and defined by the following diagram:

$$\begin{tikzcd}
\Gamma \arrow[r, "A\overline{\Gamma}^T"] & \mathcal{P}\left(T\Gamma\right) \arrow[d, "T\alpha"] \\
M \arrow[u, "\epsilon"] \arrow[ur, dashed, "A\overline{\Gamma}"] \arrow[r, "A\overline{\Gamma}^\sharp"] & \mathcal{P}\left(TM\right)
\end{tikzcd}$$

Here $\mathcal{P}\left(E\right)$ defines the power set of E, $\epsilon\left(x\right)$ the unit for an element $x \in M$ and $\alpha, \beta : \Gamma \rightrightarrows M$ denote the source and the target maps respectively. Therefore, for each $x \in M$, we have

$$\begin{aligned}
A\overline{\Gamma}_x &= A\overline{\Gamma}^T_{\epsilon(x)} \\
A\overline{\Gamma}^\sharp_x &= T_{\epsilon(x)}\alpha\left(A\overline{\Gamma}_x\right)
\end{aligned}$$

$A\overline{\Gamma}^\sharp$ is called the *base-characteristic distribution of* $\overline{\Gamma}$ and it is a generalized distribution (in the sense of Stefan and Sussmann) on M.

The relevant fact is that both distributions, $A\overline{\Gamma}^T$ and $A\overline{\Gamma}^\sharp$, are integrable (in the sense of Stefan and Sussmann), and they provide two foliations, $\overline{\mathcal{F}}$ on Γ and $\mathcal{F}$ on M.

In this paper, we have studied the properties of these foliations and obtained the following two main results:

Theorem 3.2. *Let $\Gamma \rightrightarrows M$ be a Lie groupoid and $\overline{\Gamma}$ be a subgroupoid of Γ (not necessarily a Lie groupoid) over M. Then, there exists a foliation $\overline{\mathcal{F}}$ of Γ such that $\overline{\Gamma}$ is a union of leaves of $\overline{\mathcal{F}}$.*

Theorem 3.6. *For each $x \in M$ there exists a transitive Lie subgroupoid $\overline{\Gamma}\left(\mathcal{F}\left(x\right)\right)$ of Γ with base $\mathcal{F}\left(x\right)$.*

So, although our groupoid $\overline{\Gamma}$ is not a Lie subgroupoid of Γ, we can still cover it by manifolds (leaves of the foliation $\overline{\mathcal{F}}$) and extract "transitive" and "differentiable" components (the Lie groupoids $\overline{\Gamma}\left(\mathcal{F}\left(x\right)\right) \rightrightarrows \mathcal{F}\left(x\right)$).

The next step is to apply our result to the theory a simple bodies in Continuum Mechanics. In particular, let $\mathcal{B}$ be a simple material and $\Omega\left(\mathcal{B}\right)$ its material groupoid. Then, $\Omega\left(\mathcal{B}\right)$ is not necessarily a Lie subgroupoid of $\Pi^1\left(\mathcal{B},\mathcal{B}\right)$. But, applying the results of the previous section we have that $\mathcal{B}$ can be covered by a foliation of some kind of smoothly uniform "subbodies" (these are not exactly subbodies in the usual sense of continuum mechanics [5] because of the dimension), called *material submanifolds*.

Finally, we present several examples in which the material groupoid is not a Lie subgroupoid of $\Pi^1\left(\mathcal{B},\mathcal{B}\right)$. In each case, we give explicitly the characteristic foliation which decomposes the body into smoothly uniform material submanifolds.

The paper is structured as follows: In Section 2 we give a brief introduction to (Lie) groupoids (see [6] or [7] for a detailed account of the theory of groupoids). Section 3 is devoted to develop the theory of the characteristic distribution and prove the two main theorems. In Section 4 we apply these results to continuum mechanics. Finally, some examples are discussed in Section 5.

2. Groupoids

First, we shall give a brief introduction to *Lie groupoids*. The standard reference on groupoids is [6]; for a short introduction see [7].

Definition 2.1. Let M be a set. A *groupoid* over M is given by a set Γ provided with two maps $\alpha, \beta : \Gamma \to M$ (*source* and *target maps*, respectively), $\epsilon : M \to \Gamma$ (*identities map*), $i : \Gamma \to \Gamma$ (*inversion map*) and $\cdot : \Gamma_{(2)} \to \Gamma$ (*composition law*) where