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Higgs varieties and fundamental groups

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ABSTRACT

After reviewing some "fundamental group schemes" that can be attached to a variety by means of Tannaka duality, we consider the example of the Higgs fundamental group scheme, surveying its main properties and relations with the other fundamental groups, and giving some examples.

1. Introduction

As the usual fundamental group is not well suited to study schemes equipped with the Zariski topology, Grothendieck introduced in [1] the so-called *étale fundamental group*. The usual fundamental group of a space X may be regarded as the group of deck transformations of the universal covering of X. Heuristically, one replaces covering spaces by finite étale covers; however, in this case there is no universal object, so that one needs to take an inverse limit. More technically, given a scheme X, which one assumes to be connected and locally noetherian, and after fixing a geometric point x of X, one considers the set of pairs (p, y), where $p : Y \to X$ is a finite étale cover, and $y \in Y$ is a geometric point such that $p(y) = x$. The set I of such pairs is partially ordered by the relation $(p, y) \geq (p', y')$ if there is a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "p"'] & Y' \arrow[d, "p'"] \\ & X \end{tikzcd}$$

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with $y' = f(y)$. Then one defines

$$\pi_1^{\text{ét}}(X, x) = \varprojlim_{i \in I} \operatorname{Aut}_X(p_i, y_i).$$

If X is a scheme of finite type over $\mathbb{C}$, the étale fundamental group $\pi_1^{\text{ét}}(X, x)$ is the profinite completion of the topological fundamental group $\pi_1(X, x)$.

In spite of the naturalness of its definition, the étale fundamental group, for a field of positive characteristic, fails to enjoy some quite reasonable properties; for instance, it is not a birational invariant, and is not necessarily zero for rational varieties. To circumvent these undesirable features, M.V. Nori introduced another kind of "fundamental group" (which coincides with the étale fundamental group for fields of characteristic zero) [2]. One of the properties that make this fundamental group particularly interesting is that it is introduced in terms of the so-called *Tannaka duality*. Nori considered, on a scheme X over a field $\Bbbk$, vector bundles E having the following property: there exists a Γ-torsor P on X, where Γ is a finite group, such that the pullback of E to P is trivial. Such vector bundles are said to be *essentially finite*. The category of essentially finite vector bundles on X, with the functor to the category $\mathbf{Vect}_{\Bbbk}$ of finite-dimensional vector spaces over $\Bbbk$ given by $E \mapsto E_x$, where E_x is the fibre over a fixed geometric point x of X, is an example of a *neutral Tannakian category* over $\Bbbk$. Any such category is equivalent to the category of representations of a proalgebraic group scheme over $\Bbbk$ ([3], and see Section 2.2). This group scheme is by definition Nori's fundamental group scheme $\pi_1^N(X, x)$.

Two more "fundamental groups" have been associated with a variety in terms of Tannaka duality. Langer [4,5] considered the category of numerically flat vector bundles (i.e., vector bundles that are numerically effective together with their duals). The associated fundamental group scheme was denoted by $\pi_1^S(X, x)$ (this group was introduced in the case of curves also in [6]). For varieties over the complex numbers, Simpson considered the category of semi-harmonic bundles, i.e., semistable Higgs vector bundles with vanishing rational Chern classes [7,8]. The associated fundamental group scheme was denoted $\pi_1^{\text{alg}}(X, x)$; quite interestingly, it is the proalgebraic completion of the topological fundamental group. All these fundamental groups are related by morphisms according to the scheme

$$\pi_1^{\text{alg}}(X, x) \twoheadrightarrow \pi_1^S(X, x) \twoheadrightarrow \pi_1^N(X, x) \twoheadrightarrow \pi_1^{\text{ét}}(X, x)$$

where each arrow is a faithfully flat morphism.

In [9,10] we introduced notions of numerical effectiveness and numerical flatness for Higgs bundles. The definition of these notions stems from the remark that the universal quotient bundles over the Grassmann bundles $\operatorname{Gr}_s(E)$ of a numerically effective vector bundle are numerically effective (recall that the sections of the Grassmann bundle $\operatorname{Gr}_s(E) \to X$ of a vector bundle E on X are in a one-to-one correspondence with rank s locally free quotients of E). Given a Higgs vector bundle $\mathfrak{E} = (E, \phi)$, we consider closed subschemes $\mathfrak{Gr}_s(\mathfrak{E}) \subset \operatorname{Gr}_s(E)$ that analogously parameterize locally free Higgs quotients of $\mathfrak{E}$. Then $\mathfrak{E}$ is said to be H-numerically effective if the universal Higgs quotients on $\mathfrak{Gr}_s(\mathfrak{E})$ are H-numerically effective, according to a definition which is recursive on the rank. Finally, a Higgs bundle is said to be H-numerically flat if $\mathfrak{E}$ and its dual Higgs bundle $\mathfrak{E}^*$ are H-numerically effective. H-numerically flat Higgs bundles make up again a neutral Tannakian category; the corresponding group scheme is denoted by $\pi_1^H(X, x)$ [11].

Numerically flat vector bundles, if equipped with the zero Higgs field, are H-numerically flat, so that there is a faithfully flat morphism $\pi_1^H(X, x) \twoheadrightarrow \pi_1^S(X, x)$. The relation of $\pi_1^H(X, x)$ with Simpson's proalgebraic fundamental group $\pi_1^{\text{alg}}(X, x)$ is more subtle (see Section 2.2). Again, since semi-harmonic bundles are H-numerically flat, there is a faithfully flat morphism $\pi_1^H(X, x) \twoheadrightarrow \pi_1^{\text{alg}}(X, x)$. The fact that the groups may be isomorphic is related with a conjecture about the so-called *curve semistable Higgs bundles*—i.e., Higgs bundles that are semistable after pullback to any smooth projective curve [10,12,13]. The main purpose of this note is to gather and briefly discuss what is presently known about this question.

In the preliminary Section 2 we recall the definitions and main properties of the profinite and proalgebraic completions of discrete groups, and the definition of Nori's, Langer's and Simpson's fundamental groups. Section 3 reviews the introduction of the Higgs fundamental group. The final Section 4 provides some examples.

2. Preliminaries

2.1. Profinite and proalgebraic completions

We recall here the construction of profinite and proalgebraic completions of groups. They are inverse limits of systems of finite and algebraic groups, respectively. They enjoy natural universal properties, and in both cases there is a natural homomorphism from the group to its completion with dense image. References for the two constructions are [14] and [15].

Definition 2.1. A profinite group is a topological group which is the inverse limit of an inverse system of discrete finite groups. The profinite completion $\hat{G}$ of a group G is the inverse limit of the system formed by the quotient groups G/N of G, where N are normal subgroups of G of finite index, partially ordered by inclusion.