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Analytic convergence of harmonic metrics for parabolic Higgs bundles

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ABSTRACT

In this paper we investigate the moduli space of parabolic Higgs bundles over a punctured Riemann surface with varying weights at the punctures. We show that the harmonic metric depends analytically on the weights and the stable Higgs bundle. This gives a Higgs bundle generalisation of a theorem of McOwen on the existence of hyperbolic cone metrics on a punctured surface within a given conformal class, and a generalisation of a theorem of Judge on the analytic parametrisation of these metrics.

1. Introduction

The uniformisation theorem shows that any compact Riemann surface admits a metric of constant scalar curvature within each conformal class. One way of proving this is to solve the resulting partial differential equation for the conformal factor, which was carried out by Berger when the Euler characteristic is nonpositive [1]. More generally, by solving this PDE with a different curvature function, Kazdan and Warner [2] gave necessary and sufficient conditions for a given function to be equal to the scalar curvature of some metric within a given conformal class, generalising the sufficient conditions given by Berger [3].

It was originally observed by Hitchin [4] that this PDE can be solved in the general framework of the Hitchin–Kobayashi correspondence for Higgs bundles. The theory of Hitchin [4] and Simpson [5] shows that each stable Higgs bundle admits a unique metric solving the self-duality equations. Hitchin observed in [4, Sec. 11] that, for a particular example of stable Higgs bundle (a *Fuchsian point* in the moduli space), the metric solving the self-duality equations is related to a metric solving the constant scalar curvature equations on the underlying compact Riemann surface. Moreover, by deforming the Higgs field one can obtain all of the constant curvature metrics on the underlying smooth surface, which leads to Hitchin's construction of Teichmüller space for genus $g \geq 2$. The key idea is to find a stable Higgs bundle for which the Hitchin–Kobayashi correspondence produces the required metric, instead of solving the PDE for the metric directly.

Subsequently Simpson [6] showed that a stable parabolic Higgs bundle with regular singularities admits a unique metric solving the self-duality equations on a punctured surface. An important aspect of the theory for noncompact surfaces is the need to control the growth of the metric near the punctures. These growth conditions are determined by the stability condition in the form of weights (introduced by Mehta and Seshadri [7] for parabolic bundles without a Higgs field) and by the weight filtration, which depends on the residue of the Higgs field at the punctures. Biswas, Arés-Gastesi and Govindarajan [8]

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showed that Simpson's theory for parabolic Higgs bundles can be used to produce constant curvature cusp metrics on punctured surfaces, and they extended this construction to the higher Teichmüller theory introduced by Hitchin [9].

With respect to metrics with a conical singularity at the punctures, results of McOwen [10] prove the existence of hyperbolic cone metrics of any cone angle within a given conformal class, and subsequent work of Judge [11] shows that these metrics depend analytically on the cone angles. Both of these proofs are in the spirit of the approach of Berger [1,3] and Kazdan–Warner [2] which involves solving a PDE for the conformal factor. Troyanov [12] also used this approach to give necessary and sufficient conditions for a function to be the curvature of a metric with conical singularities within a given conformal class. When the cone angles are of the form $\frac{2\pi}{m}$ for $m \in \mathbb{Z}$, a Higgs bundle approach to constructing cone metrics was developed by Nasatyr and Steer [13], who studied orbifold Higgs bundles on a finite ramified cover of the underlying compact Riemann surface. Nasatyr and Steer also proved a connection between orbifold Higgs bundles and parabolic Higgs bundles with rational weights and trivial weight filtration.

In this paper we investigate parabolic Higgs bundles with varying weights and study the dependence of the harmonic metric on the weight and the Higgs bundle. The following theorem is the main result, which uses the Hitchin–Kobayashi correspondence for parabolic Higgs bundles ([6, Thm. 6]) to prove a Higgs bundle generalisation of Judge's theorem.

Theorem 1.1 (*Theorem 4.1*). *For an initial stable parabolic Higgs bundle, the metric solving the self-duality equations depends analytically on the choice of weights and stable Higgs bundle in a neighbourhood of the initial weight and Higgs bundle.*

Judge's theorem then appears in the case of a fixed Higgs bundle given by a Fuchsian point in the rank 2 moduli space (Corollary 4.2). Along the way we obtain a new Higgs bundle proof of McOwen's theorem (Corollary 3.3), thus generalising the results of Nasatyr and Steer to metrics with arbitrary cone angles.

Organisation of the paper. In Section 2 we recall the necessary definitions and results for parabolic Higgs bundles from [7] and [6] and recall the necessary results on weighted Sobolev spaces from [14] that will be used in the proof of the main theorem. In Section 3 we define a model rank 2 harmonic bundle on a punctured disk associated to a given cone angle θ. The holonomy of the associated flat connection around the puncture corresponds to an elliptic element of $\mathrm{SL}(2, \mathbb{C})$. As the cone angle converges to zero we show that the harmonic cone metrics converge to the cusp metric studied by Simpson [6, Sec. 5] and that (modulo gauge) the corresponding sequence of holonomy representations given by elliptic elements of $\mathrm{SL}(2, \mathbb{C})$ converges to a parabolic element of $\mathrm{SL}(2, \mathbb{C})$. Applying Simpson's nonabelian Hodge theorem [6] then gives us a new proof of McOwen's theorem (Corollary 3.3).

In Section 4 we globalise the results of Section 3 and show in Theorem 4.1 that for an initial algebraically stable parabolic Higgs bundle, the harmonic metric on E depends analytically on the weights and stable Higgs bundle. In particular, as a sequence of weights converges to a fixed weight, then the harmonic metrics, flat connections and holonomy representations converge. Restricting to the rank 2 case gives a new Higgs bundle proof of Judge's theorem [11].

2. Background and definitions

2.1. Parabolic bundles and model metrics

In this section we recall some basic notions of parabolic vector bundles and Higgs bundles from [7] and [6] which are relevant to the rest of the paper. Since we use the results of Simpson from [6] throughout the rest of the paper, then we will also follow the terminology and notation from [6] in Definitions 2.1–2.4.

Let $\bar{X}$ be a compact Riemann surface with marked points $\{p_1, \dots, p_n\}$ and let $X = \bar{X} \setminus \{p_1, \dots, p_n\}$. Let $i : X \to \bar{X}$ denote the inclusion. We will use a Riemannian metric on $\bar{X}$ in the conformal class determined by the complex structure to define the distance from each marked point. For simplicity, in the sequel we will state the definitions for the case of one marked point p; the general case follows in exactly the same way.

Let $E \to X$ be a holomorphic vector bundle. The notion of a filtered bundle from [6] involves a choice of extension of E across the puncture p.

Definition 2.1 (*Filtered Regular Higgs Bundle*). A *filtered vector bundle* is an algebraic vector bundle $E \to X$ together with a one-parameter family of vector bundles $E_\alpha \to \bar{X}$ indexed by $\alpha \in \mathbb{R}$ such that $E = i^* E_\alpha$ for all α and

- E_α is a subsheaf of E_β for each $\alpha \geq \beta$,
- for each α there exists $\varepsilon' > 0$ such that $E_{\alpha-\varepsilon} = E_\alpha$ for all $0 < \varepsilon < \varepsilon'$, and
- $E_{\alpha+1} = E_\alpha[-p]$ for all α.

A *filtered regular Higgs bundle* $(E, \phi, \{E_\alpha\})$ is a filtered vector bundle $(E, \{E_\alpha\})$ together with a section $\phi \in H^0(\mathrm{End}(E_0) \otimes K_{\bar{X}}[p])$ such that ϕ preserves the subsheaf $E_\alpha \subset E_0$ for each $\alpha \in (0, 1]$.

The equivalence of this definition with the definition of a parabolic structure from [7] is given as follows. Given a filtered bundle $\{E_\alpha\}_{\alpha \in \mathbb{R}}$, let $E_{p,0}$ denote the fibre of $E_0 \to \bar{X}$ over $p \in \bar{X}$. Then the vector space $E_{p,0}$ has an induced filtration $\{E_{p,\alpha}\}$ indexed by $0 \leq \alpha < 1$. For each α, define $\mathrm{Gr}_\alpha(E_{p,0})$ to be the direct limit of the system $E_{p,\alpha}/E_{p,\beta}$ over all $\beta > \alpha$. The *weights* of the parabolic structure are the values of α in $[0, 1)$ such that $\dim_{\mathbb{C}} \mathrm{Gr}_\alpha(E_{p,0}) > 0$. In the sequel we will use α or β to denote the weights of a given parabolic structure, and μ or ν to denote the set $\{\alpha_1, \dots, \alpha_n\}$ of weights counted with multiplicity.