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Deformations of infinite-dimensional Lie algebras, exotic cohomology, and integrable nonlinear partial differential equations

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ABSTRACT

The important unsolved problem in theory of integrable systems is to find conditions guaranteeing existence of a Lax representation for a given PDE. The exotic cohomology of the symmetry algebras opens a way to formulate such conditions in internal terms of the PDEs under the study. In this paper we consider certain examples of infinite-dimensional Lie algebras with nontrivial second exotic cohomology groups and show that the Maurer–Cartan forms of the associated extensions of these Lie algebras generate Lax representations for integrable systems, both known and new ones.

1. Introduction

The existence of a Lax representation is the key property of integrable equations, [1,2], and a starting setting for a number of techniques to study nonlinear partial differential equations (PDEs) such as Bäcklund transformations, nonlocal symmetries and conservation laws, recursion operators, Darboux transformations, etc. Although these structures are of great significance in the theory of integrable PDEs, up to now the problem of finding conditions for a PDE to admit a Lax representation is open. In [3] we propose an approach for solving this problem in internal terms of the PDE under the study. We show there that for some PDEs their Lax representations can be derived from the second exotic[1] cohomology of the symmetry pseudogroups of the PDEs. The main advantage of this approach is that it allows one to get rid of apriori assumptions about the defining equations of the Lax representation. In this paper we generalize the constructions of [3]. We consider a deformation of the tensor product of the Lie algebra of vector fields on a line and the algebra of truncated polynomials as well as certain extensions of this deformation and show that at some values of the deformation parameter the Maurer–Cartan forms of the obtained Lie algebras produce Lax representations for some known as well as some new integrable systems.

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[1] Unlike in [3], in this paper we follow [4] and use the term "exotic cohomology" instead of "deformed cohomology", since here we discuss deformations of Lie algebras which are not related to "deformed cohomology" in the sense of [3].

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2. Preliminaries

All considerations in this paper are local. All functions are assumed to be real-analytic.

2.1. Coverings of PDEs

The coherent geometric formulation of Lax representations, Wahlquist–Estabrook prolongation structures, Bäcklund transformations, recursion operators, nonlocal symmetries, and nonlocal conservation laws is based on the concept of differential covering of a PDE [5,6]. In this subsection we closely follow [7,8] to present the basic notions of the theory of differential coverings.

Let $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $\pi : (x^1, \dots, x^n, u^1, \dots, u^m) \mapsto (x^1, \dots, x^n)$ be a trivial bundle, and $J^\infty(\pi)$ be the bundle of its jets of the infinite order. The local coordinates on $J^\infty(\pi)$ are $(x^i, u^\alpha, u^\alpha_I)$, where $I = (i_1, \dots, i_n)$ is a multi-index, and for every local section $f : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$ of π the corresponding infinite jet $j_\infty(f)$ is a section $j_\infty(f) : \mathbb{R}^n \to J^\infty(\pi)$ such that $u^\alpha_I(j_\infty(f)) = \frac{\partial^{\#I} f^\alpha}{\partial x^I} = \frac{\partial^{i_1+\dots+i_n} f^\alpha}{(\partial x^1)^{i_1} \dots (\partial x^n)^{i_n}}$. We put $u^\alpha = u^\alpha_{(0,\dots,0)}$. Also, in the case of $m = 1$ and, e.g., $n = 4$ we denote $x^1 = t$, $x^2 = x$, $x^3 = y$, $x^4 = z$, and $u^1_{(i,j,k,l)} = u_{t\dots tx\dots xy\dots yz\dots z}$ with i times t, j times x, k times y, and l times z.

The vector fields

$$D_{x^k} = \frac{\partial}{\partial x^k} + \sum_{\#I \ge 0} \sum_{\alpha=1}^{m} u^\alpha_{I+1_k} \frac{\partial}{\partial u^\alpha_I}, \qquad k \in \{1, \dots, n\},$$

with $I + 1_k = (i_1, \dots, i_k, \dots, i_n) + 1_k = (i_1, \dots, i_k + 1, \dots, i_n)$ are referred to as *total derivatives*. They commute everywhere on $J^\infty(\pi)$: $[D_{x^i}, D_{x^j}] = 0$.

A system of PDEs $F_r(x^i, u^\alpha_I) = 0$, $\#I \le s$, $r \in \{1, \dots, \sigma\}$, of the order $s \ge 1$ with $\sigma \ge 1$ defines the submanifold $\mathcal{E} = \{(x^i, u^\alpha_I) \in J^\infty(\pi) \mid D_K(F_r(x^i, u^\alpha_I)) = 0,\ \ \#K \ge 0\}$ in $J^\infty(\pi)$.

Denote $\mathcal{W} = \mathbb{R}^\infty$ with coordinates w^a, $a \in \mathbb{N} \cup \{0\}$. Locally, an (infinite-dimensional) *differential covering* over $\mathcal{E}$ is a trivial bundle $\tau : J^\infty(\pi) \times \mathcal{W} \to J^\infty(\pi)$ equipped with the *extended total derivatives*

$$\tilde{D}_{x^k} = D_{x^k} + \sum_{a=0}^{\infty} T^a_k(x^i, u^\alpha_I, w^b) \frac{\partial}{\partial w^a} \tag{1}$$

such that $[\tilde{D}_{x^i}, \tilde{D}_{x^j}] = 0$ for all $i \ne j$ whenever $(x^i, u^\alpha_I) \in \mathcal{E}$. For the partial derivatives of w^a which are defined as $w^a_{x^k} = \tilde{D}_{x^k}(w^a)$ we have the system of *covering equations*

$$w^a_{x^k} = T^a_k(x^i, u^\alpha_I, w^b).$$

This over-determined system of PDEs is compatible whenever $(x^i, u^\alpha_I) \in \mathcal{E}$.

Dually the covering with extended total derivatives (1) is defined by the differential ideal generated by the *Wahlquist–Estabrook forms*, [2, p. 81],

$$\varpi^a = dw^a - \sum_{k=1}^{n} T^a_k(x^i, u^\alpha_I, w^b)\, dx^k.$$

This ideal is integrable on $\mathcal{E}$, that is,

$$d\varpi^a \equiv \sum_b \eta^a_b \wedge \varpi^b \mod \langle \vartheta_I \rangle,$$

where η^a_b are some 1-forms on $\mathcal{E} \times \mathcal{W}$ and $\vartheta_I = (du^\alpha_I - \sum_k u^\alpha_{I+1_k} dx^k)|_{\mathcal{E}}$.

2.2. Exotic cohomology

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$ and $\rho : \mathfrak{g} \to \mathrm{End}(V)$ be its representation. Let $C^k(\mathfrak{g}, V) = \mathrm{Hom}(\Lambda^k(\mathfrak{g}), V)$, $k \ge 1$, be the space of all k-linear skew-symmetric mappings from $\mathfrak{g}$ to V. Then the Chevalley–Eilenberg differential complex

$$V = C^0(\mathfrak{g}, V) \xrightarrow{d} C^1(\mathfrak{g}, V) \xrightarrow{d} \cdots \xrightarrow{d} C^k(\mathfrak{g}, V) \xrightarrow{d} C^{k+1}(\mathfrak{g}, V) \xrightarrow{d} \cdots$$

is generated by the differential defined by the formula

$$\begin{aligned} d\theta(X_1, \dots, X_{k+1}) &= \sum_{q=1}^{k+1} (-1)^{q+1} \rho(X_q)(\theta(X_1, \dots, \hat{X}_q, \dots, X_{k+1})) \\ &+ \sum_{1 \le p < q \le k+1} (-1)^{p+q} \theta([X_p, X_q], X_1, \dots, \hat{X}_p, \dots, \hat{X}_q, \dots, X_{k+1}). \end{aligned} \tag{2}$$