Journal of Geometry and Physics 128 (2018) 169–184

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys

Hybrid normed ideal perturbations of n-tuples of operators I

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ARTICLE INFO

Article history:
Received 21 December 2017
Accepted 16 February 2018
Available online 26 February 2018

Dedicated to Alain Connes on the occasion of his 70th birthday.

MSC:
primary 47L30
secondary 47L20
47A13
42B20

Keywords:
Hybrid normed ideal perturbation
Modulus of quasicentral approximation
Voiculescu non-commutative Weyl–von Neumann theorem
Mixed homogeneity singular integral
Absolutely continuous n-dimensional spectral measure

ABSTRACT

In hybrid normed ideal perturbations of n-tuples of operators, the normed ideal is allowed to vary with the component operators. We begin extending to this setting the machinery we developed for normed ideal perturbations based on the modulus of quasicentral approximation and an adaptation of our non-commutative generalization of the Weyl–von Neumann theorem. For commuting n-tuples of hermitian operators, the modulus of quasicentral approximation remains essentially the same when $\mathcal{C}_n^-$ is replaced by a hybrid n-tuple $\mathcal{C}_{p_1,\dots}^-, \dots, \mathcal{C}_{p_n}^-, p_1^{-1} + \cdots + p_n^{-1} = 1$. The proof involves singular integrals of mixed homogeneity.

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1. Introduction

In [1] we adapted the Voiculescu non-commutative Weyl–von Neumann type theorem [2] to general normed ideals, which provided a new approach to normed ideal perturbation questions for n-tuples of operators on Hilbert space (see [3] for a survey). A key quantity in this approach is the numerical invariant $k_\Phi(\tau)$, the modulus of quasicentral approximation where Φ is the norming function of a normed ideal $\mathcal{G}_\Phi^{(0)}$ (see [4] or [5]) and $\tau = (T_1, \dots, T_n)$ is an n-tuple of operators. In particular, this technique has been successful in dealing with generalizations to commuting n-tuples of hermitian operators of the one hermitian operator Kuroda–Weyl–von Neumann diagonalizability results and of the Kato–Rosenblum result on preservation of the Lebesgue absolutely continuous part under trace class perturbation [1,6–8]. Since normed ideals provide the infinitesimals in non-commutative geometry there have also been technical uses of the machinery in non-commutative geometry [9–12].

The present paper, the first in a series, is the beginning of an extension of our approach to hybrid normed ideal perturbations. This means that instead of two n-tuples $\tau = (T_j)_{1\le j\le n}$, $\tau' = (T'_j)_{1\le j\le n}$ so that $T_j - T'_j \in \mathcal{G}_\Phi^{(0)}$ for $1 \le j \le n$, we will consider now the more general situation $T_j - T'_j \in \mathcal{G}_{\Phi_j}^{(0)}$ for $1 \le j \le n$, that is for each index j there is now a different normed ideal $\mathcal{G}_{\Phi_j}^{(0)}$ $(1 \le j \le n)$. Of course, much of the basics easily extends to the hybrid setting, with proofs that are almost verbatim repetitions of the same-ideal proofs. However, once we get to hybrid perturbations of n-tuples of commuting

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https://doi.org/10.1016/j.geomphys.2018.02.016
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hermitian operators there is an unexpected new phenomenon: the threshold ideal $\mathcal{C}_n^-$ (aka $\mathcal{C}_{n,1}$ on the Lorentz scale) can be replaced by any hybrid n-tuple of normed ideals $(\mathcal{C}_{p_1}^-, \ldots, \mathcal{C}_{p_n}^-)$ where $\sum_{1\le j\le n} p_j^{-1} = 1$, $1 < p_j$ for $1 \le j \le n$. Indeed the generalized singular and absolutely continuous subspaces with respect to $(\mathcal{C}_{p_1}^-, \ldots, \mathcal{C}_{p_n}^-)$ like in the case of $\mathcal{C}_n^-$ coincide with the Lebesgue singular and Lebesgue absolutely continuous parts of the n-dimensional spectral measure. Actually even more, up to a factor of proportionality, the modulus of quasicentral approximation, is given by the same formula for all hybrid choices $(\mathcal{C}_{p_1}^-, \ldots, \mathcal{C}_{p_n}^-)$ with $\sum_{1\le j\le n} p_j^{-1} = 1$, which also includes the case when $p_1 = \cdots = p_n = n$. While the general lines of the proofs are similar, one must refine some of the analysis involved with the singular integral underlying the non-vanishing of the modulus of quasicentral approximation, especially since one must replace the homogeneous singular integral with a new one with mixed homogeneity.

In addition to this introduction which is Section 1 and to the references, there are ten more sections.

Section 2 summarizes for the reader's convenience notation and definitions about normed ideals which we shall use.

Section 3 introduces the modulus of quasicentral approximation $k_\varphi(\tau)$ in the hybrid setting. Many of the simplest facts about $k_\varphi(\tau)$ are straightforward generalizations of [1] and [8] and the proofs could be omitted.

Section 4 deals with traces of sums of commutators, more precisely the commutators are between the operators in a given n-tuple and operators from the duals of the normed ideals. Traces of sums of such commutators are the source of lower bounds for the modulus of quasicentral approximation. This section is a straightforward generalization to the hybrid setting of results in [13].

Section 5 contains the hybrid version of the adaptation to normed ideals in [1] of our main result in [2] (see also [14]). Again the proofs follow closely those in [1] and were omitted.

Section 6 gives the hybrid generalization of the decomposition of the Hilbert space into Φ-absolutely continuous and Φ-singular subspaces with respect to an n-tuple of operators τ in [8]. The straightforward generalizations of the proofs are omitted. In addition we provide also a strengthening of one of the results.

Section 7 deals with upper bounds for k_φ and diagonalization mod φ results for n-tuples of commuting hermitian operators where φ is a hybrid n-tuple of normed ideals. Upper bounds for k_φ are much easier than lower bounds and this is why we can deal with larger classes of hybrid normed ideal perturbations and results are sharper. We obtain also diagonalization results in the case of spectral measures which are singular with respect to Lebesgue measure. These are, of course, the automatic consequence of vanishing of k_φ results. The results we get in the case of $n = 2$, where the pair of norming functions are in duality, provide a large class of hybrid examples.

Sections 8 and 9 deal with the estimate of the singular integral with mixed homogeneity which we need for the lower bound of k_φ of an n-tuple of commuting hermitian operators. The general outline is similar to [7], but we need to refine some of the details and to use some stronger facts about absolutely convergent Fourier series.

In Section 10, based on the previous technical facts, we prove the formula for $k_\varphi(\tau)$ where τ is an n-tuple of commuting hermitian operators and φ a hybrid n-tuple of normed ideals $(\mathcal{C}_{p_1}^-, \ldots, \mathcal{C}_{p_n}^-)$ with $p_1^{-1} + \cdots + p_n^{-1} = 1$ and $1 \le p_j$, $1 \le j \le n$. In particular, this also implies that the φ-singular and φ-absolutely continuous subspaces are the same as the Lebesgue-singular and Lebesgue-absolutely continuous subspaces for the spectral measure of τ.

In Section 11 we show that the commutants mod normed ideals have a natural generalization to the hybrid setting and we show that the relation between the modulus of quasicentral approximation and approximate units for the compact ideal [15] carries over to this more general setting.

2. Preliminaries

Most of this section is about the notation we will use for operators and normed ideals of operators [4,5].

Let $\mathcal{H}$ be a separable infinite-dimensional complex Hilbert space. By $\mathcal{L}(\mathcal{H})$, $\mathcal{L}_+(\mathcal{H})$, $\mathcal{K}(\mathcal{H})$, $\mathcal{P}(\mathcal{H})$, $\mathcal{R}(\mathcal{H})$, $\mathcal{R}_1^+(\mathcal{H})$ (or when $\mathcal{H}$ is not in doubt simply $\mathcal{L}$, $\mathcal{L}_+$, $\mathcal{K}$, $\mathcal{P}$, $\mathcal{R}$, $\mathcal{R}_1^+$) we shall denote respectively the bounded operators on $\mathcal{H}$, the positive bounded operators on $\mathcal{H}$, the positive bounded operators on $\mathcal{H}$, the compact operators, the finite rank hermitian projections, the finite rank operators and the finite rank positive contractions.

Let $\hat{c}$ be the space of sequence $(\xi_j)_{j\in\mathbb{N}}$, $\xi_j \in \mathbb{R}$ with finite support. A norming function Φ is a function on $\hat{c}$ taking values in $\mathbb{R}$ so that

I. $\xi \neq 0 \Rightarrow \Phi(\xi) > 0$

II. $\Phi(\alpha\xi) = |\alpha|\Phi(\xi),\ \alpha \in \mathbb{R}$

III. $\Phi(\xi + \eta) \le \Phi(\xi) + \Phi(\eta)$

IV. $\Phi((1, 0, 0, \ldots)) = 1$

V. $\Phi((\xi_j)_{j\in\mathbb{N}}) = \Phi((\xi_{\pi(j)})_{j\in\mathbb{N}})$ if π is a permutation of $\mathbb{N}$.

The set of norming functions will be denoted by $\mathcal{F}$.

If $T \in \mathcal{R}(\mathcal{H})$ and $\Phi \in \mathcal{F}$ then

$$|T|_\Phi = \Phi((s_j)_{j\in\mathbb{N}})$$

where $(s_j)_{n\in\mathbb{N}}$ are the eigenvalues of $(T^*T)^{1/2}$ (multiple eigenvalues repeated according to multiplicity).

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