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Nonlocal integrable PDEs from hierarchies of symmetry laws: The example of Pohlmeyer–Lund–Regge equation and its reflectionless potential solutions

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ABSTRACT

By following the ideas presented by Fukumoto and Miyajima in Fukumoto and Miyajima (1996) we derive a generalized method for constructing integrable nonlocal equations starting from any bi-Hamiltonian hierarchy supplied with a recursion operator. This construction provides the right framework for the application of the full machinery of the inverse scattering transform. We pay attention to the Pohlmeyer–Lund–Regge equation coming from the nonlinear Schrödinger hierarchy and construct the formula for the reflectionless potential solutions which are generalizations of multi-solitons. Some explicit examples are discussed.

1. Introduction

The hierarchies of bi-Hamiltonian PDEs are sets of commuting evolution equations which can be constructed recursively [1]. Their commutation implies that such flows can be summed preserving the integrability property. Historically, the relevance of commuting flow summations in hydrodynamics goes back to the paper by Fukumoto and Miyazaki written in 1991 [2], where the authors connect the vortex motion in a three dimensional Euler fluid to the Hirota equation, which is the sum of the nonlinear Schrödinger (NLS) and complex modified Korteweg–de Vries Hamiltonian flows. Along the same line of research, in 1996 Fukumoto and Miyajima [3] found an interesting connection between the NLS hierarchy and the Pohlmeyer-Lund-Regge (PLR) equation: the PLR equation can be obtained as a suitable infinite sum of commuting flows in the NLS hierarchy. This property, *de facto*, is a Hamiltonian proof of the integrability of the PLR equation. In this paper we generalize such construction to any bi-Hamiltonian hierarchy for which can be defined a recursion operator formally inverting one of the Poisson bi-vectors (see e.g. ωN-manifolds in [4]). The result of these infinite summation methods is generically a nonlocal PDE. This construction could appear as an academic exercise but it is an explicit way to construct nonlocal integrable systems whose interest is growing. Moreover, in the inverse scattering transform (IST) framework, it can be used to explicitly find reflectionless solutions. We concretely illustrate the method in the case of soliton-like solutions of the PLR equation.

The PLR equation in a uniform static external field has been proposed in 1976 by Lund and Regge [5] as a possible model describing both motion of extended relativistic strings and (in a particular limit) nonrelativistic vortices in superfluids. It is

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explicitly given by

$$\begin{aligned} &\mathbf{X}_{tt} - \mathbf{X}_{xx} = -2\mathbf{X}_t \times \mathbf{X}_x \,, \\ &{\mathbf{X}_t}^2 + {\mathbf{X}_x}^2 = 1 \,, \\ &\mathbf{X}_t \cdot \mathbf{X}_x = 0, \end{aligned} \tag{1.1}$$

where $\mathbf{X} \in \mathbb{R}^3$ is the vector of coordinates of the string. In the same year Pohlmeyer [6] proposed the same equation in the framework of Hamiltonian systems as an integrable generalization of the sine-Gordon equation. The ubiquity of the PLR equations as a model of very different phenomena involves also plasma physics: in a relatively recent paper [7], Schief proved a relation between a particular constrained version of the PLR equation and magnetohydrodynamics.

An interesting mathematical property of the PLR equation (see e.g. [3]) is that, through the Hasimoto map $q = K \exp(\int^x \tau \, dx)$, K and τ being the curvature and torsion of the curve $\mathbf{X}$, respectively, Eq. (1.1) becomes

$$iq_t - \varepsilon q_{xt} + 2\varepsilon kq \int^x |q|_t^2 dx = -q_{xx} + 2k|q|^2 q \,. \tag{1.2}$$

This form of the PLR equation, used throughout the paper, can be viewed as a nonlocal generalization of the NLS equation. Another equation which shares with PLR a similar property, i.e. to be a nonlocal and integrable generalization of NLS, is the Landau–Lifshitz equation [8].

In our paper we derive an explicit multisoliton solution formula for the PLR equation (1.2). In fact, to the best of our knowledge only few examples of soliton (see e.g. [9]) or shape invariant [3] solutions have been obtained for Eq. (1.2) and typically in implicit form.

To get such results we generalize the procedure used in [10] to solve the Hirota equation. This procedure combines the so-called matrix triplet method (which is partially based on the IST; see Section 3 for more details) with the observation that if V_ε denotes the time evolution matrix for the summed flows of the NLS hierarchy, we have

$$V_\varepsilon = \sum_{n=1}^{\infty} \varepsilon^{n-1} V_n, \tag{1.3}$$

where V_n is the evolution matrix of the nth flow of the hierarchy and ε is a small positive parameter. For the class of reflectionless solutions we prove that asymptotically the series (1.3) is absolutely convergent. Moreover, even though the problem of establishing the absolute convergence of the series (1.3) has so far remained unanswered for the class of non reflectionless solutions, it is interesting to observe that the summations truncated after the first N flows give an interesting indication of the qualitative features of the flow summations on a hierarchy and an approximation of the PLR solutions. In fact, as suggested in a purely physical context by the study of the axial velocity in vortex filaments [10], the main effect is the variation of the typical speed of the solution for fixed amplitudes.

To obtain an explicit manageable formula we use the so-called matrix triplet method: this method is based on the observation that the integral kernel of the Marchenko integral equation has separated variables if the reflection coefficient vanishes identically. In that case there exists a triplet of matrices (A, B, C), of sizes $p \times p$, $p \times 1$, and $1 \times p$, such that the Marchenko kernel is given by

$$\Omega_l(x + y, t) = Ce^{tH} e^{-(x+y)A} B,$$

where the $p \times p$ matrices A and H commute and A has only eigenvalues with positive real part. Usually H is a simple function of A. Solving the Marchenko equation by elementary means, we arrive at the solution of the initial-value problem in terms of the matrix triplet (A, B, C) and the matrix H describing the time dependence. The expression obtained can then be written in terms of elementary functions using computer algebra. The matrix triplet method has been applied successfully to the KdV equation [11], the focusing NLS equation [12–16], the sine-Gordon equation [17,18], the modified Korteweg–de Vries (mKdV) equation [19], the Hirota equation [10], and the Heisenberg ferromagnetic equation [20]. In this article we show how to get a solution of the Marchenko equation associated to the PLR equation (1.2).

2. Bi-Hamiltonian structures and nonlocal integrable equations

In this section we present a generalization of the method used in [3] for the construction of the PLR equation. Using the classical bi-Hamiltonian recursion relations, it is possible to construct a nonlocal integrable equation associated to the hierarchy by assuming the existence of a recursion operator. In this framework this requirement is always fulfilled when one of the two Poisson structures is the inverse of a symplectic structure. In infinite dimensional spaces the notion of invertibility of a tensor boils down to a suitable choice of the function space in which the theory is formulated. Actually, the Poisson tensors are differential operators acting on the variation of Hamiltonian functionals. The study of this problem, also in well-known cases such as constant structures, goes beyond the scope of this paper: we refer the reader to the excellent classical paper by Maltsev and Novikov [21], where the NLS case is one of the many cases considered.