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Coulomb dynamics of three equal negative charges in field of fixed two equal positive charges

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ABSTRACT

Quasi-periodic solutions of the Coulomb equation of motion for three identical negative charges in the field of two fixed point positive charges are found. The center Lyapunov theorem is applied to the transformed equation of motion.

1. Introduction

It is the important task of mathematics to find solutions of newtonian equations of motion of manybody d-dimensional mechanical systems represented in terms of series converging on the infinite time interval. These equations for N-body systems with a potential energy U and masses $m_j, j = 1, \dots, N$ look like

$$m_j \frac{d^2x_j}{dt^2} = -\frac{\partial U(x_{(N)})}{\partial x_j}, \qquad j = 1, \dots, N, \quad x_{(N)} = (x_1, \dots, x_N) \in \mathbb{R}^{dN}, x_j = (x_j^1, \dots, x_j^d). \tag{1.1}$$

The existence of an equilibrium of the potential energy makes it possible to find such solutions with the help of fundamental theorems among which is the Lyapunov center theorem.

The Coulomb systems of two and three negative equal charges e_0 in the field of fixed two equal positive charges e' have equilibrium configurations [1–3]. This fundamental fact allowed us to construct periodic [1,2], bounded [1,2] and quasi-periodic [3] solutions close to the equilibria of the Coulomb equation of motion for the negative charges. For the line systems we applied the Weinstein [4,5], Moser [6] and center Lyapunov [7–12] theorems which demand a knowledge of the spectra of the matrix U^0 of second derivatives of the potential energy at the equilibrium(for equal masses). The last two theorems, guaranteeing the existence of the periodic solutions in terms of convergent series, restrict the values of $\frac{e_0}{e'}$ through a non-resonance condition. The Weinstein theorem establishes the existence of the periodic solutions without this condition but can be applied only for mechanical systems with a stable equilibrium (U^0 is positive definite and the equilibrium is a minimum of the potential energy). The construction of the bounded solutions in [2] for the line and planar systems does not demand the non-resonant condition to be true and is based on the generalization of the semi-linearization Siegel technique exposed in the section Lyapunov theorem in [7]. Periodic solutions are also found in planar Coulomb systems of $n-1, n > 2$ equal negative charges and one and three positive charges [13,14]. These solutions describe positions of the negative charges in the form of regular polygons.

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The dynamics in the Coulomb system of a negative charge in a field of many fixed positive charges can be deduced from [15–17].

Mechanical systems with an integral of motion and an equilibrium have U^0 characterized by the zero eigenvalue [7]. This does not allow one to find directly the solutions of their equations of motion on the infinite time interval with the help of the mentioned theorems.

The mentioned space systems of two and three equal negative charges possess a rotational symmetry, its angular momentum Q is an integral of motion and there is the continuum of equilibria parametrized by points of a circle centered at the origin. The matrices of second derivatives of the potential energy at the equilibria are similar to the simplest matrix U^0 which corresponds to the equilibrium determined by the three equal negative charges located at a coordinate axis and the two fixed positive charges located at an equal distance from the origin at another coordinate axis.

In this paper we find quasi-periodic solutions of the Coulomb equation of motion whose Euclidean norms are periodic functions for the system of three equal negative charges in the field of two fixed positive charges. This result is an analog of the result of [3] and a consequence of the representation of U^0 as the direct sum of three three-dimensional matrices whose eigenvalues are found explicitly.

We circumvent the obstruction of the zero eigenvalue as in [3] with the help of the procedure of the elimination of node from the celestial mechanics (section 18 in [7]). The main idea of this procedure is to produce a canonical transformation turning the integrals of motions into cyclic variables (a transformed Hamiltonian does not depend on them). Then linear part of the equation of motion for them will be zero and the canonical matrix, i.e the matrix generating the linear part of the equation of motion, for remaining variables will not contain the zero eigenvalue (see the Appendix). The procedure of the elimination of node is described in the following theorem (the proof of its first two statements are given in [7]).

Theorem 1.1. *Let $H(x,p)$ be a $2n$-dimensional Hamiltonian, Q be its time independent integral and $w(u,p)$ be a generating function of the canonical transformation such that*

$$v_k = \frac{\partial w}{\partial u_k}, \qquad x_k = \frac{\partial w}{\partial p_k}, \qquad k = 1, \dots, n, \tag{1.2}$$

$$\frac{\partial w}{\partial u_n} = Q(x,p), \qquad W_{k,j} = \frac{\partial^2 w}{\partial u_k \partial p_j}, \qquad DetW \neq 0. \tag{1.3}$$

Then the transformed Hamiltonian $H'(u,v)$ does not depend on u_n. Let also the canonical matrix of H have doubly degenerate zero eigenvalue, this canonical transformation and the Hamiltonian be holomorphic at a neighborhood of the equilibrium. Then the canonical matrix of the $2(n-1)$-dimensional Hamiltonian equation

$$\dot{u}_j = \frac{\partial H'}{\partial v_j}, \qquad \dot{v}_j = -\frac{\partial H'}{\partial u_j}, \qquad j = 1, \dots, n-1, \tag{1.4}$$

does not have the zero eigenvalue for the equilibrium value Q^0 of Q and eigenvalues of the canonical matrices of H and H' are identical.

A separation of cyclic variables, generated by integrals of motion, in a Hamiltonian equation is also described in [18].

We find w as

$$w = \sum_{j=1}^{n} g_k(u_1, \dots, u_n) p_k,$$

where $n = 9$, for a special numeration of charge coordinates and momenta $(x_j; p_j) = (x_j^\alpha; p_j^\alpha), j = 1,2,3, \alpha \leq 3$. Such the representation for w, the equation for g_k and its solution is inspired by the Celestial Mechanics [7]. As a result solutions of the Coulomb equation are given by

$$x_j^\alpha(t) = \sum_{k=1}^{n-1} u_k(t)[\gamma_{k,j,\alpha} + \gamma'_{k,j,\alpha}\cos(u_n(t)) + \gamma''_{k,j,\alpha}\sin(u_n(t))], \tag{1.5}$$

where $\gamma, \gamma', \gamma''$ are constants and $u_k(t), k = 1, \dots, n$ are solutions of the equation with the Hamiltonian H'. We show with the help of the center Lyapunov theorem that (1.4) for our system possesses periodic solutions.

Let $u_{(n-1)}, v_{(n-1)}$ be a periodic solution of (1.4) with a period τ. Then

$$(\frac{\partial H'}{\partial v_n})(u_{(n-1)}, v_{(n-1)}, Q^0) = H'_{n-1}(t)$$

is also a periodic function such that

$$H'_{n-1}(t) = H^0_{n-1}(t) + \xi, \qquad \int_t^{t+\tau} H^0_{n-1}(s)ds = 0,$$