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Volume growth and puncture repair in conformal geometry

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ABSTRACT

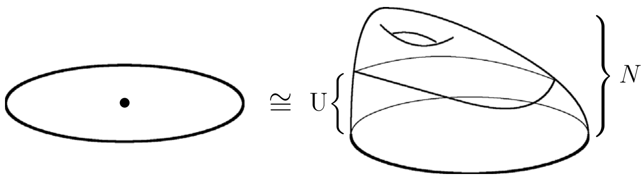
Suppose M is a compact Riemannian manifold and $p \in M$ an arbitrary point. We employ estimates on the volume growth around p to prove that the only conformal compactification of $M \setminus \{p\}$ is M itself.

1. Introduction

Though this article is primarily concerned with conformal differential geometry in dimension ≥ 3, the phenomenon we wish to describe also occurs in dimension 2 as follows.

Theorem 1. *Suppose that M is a compact connected Riemann surface and $p \in M$. Suppose that N is a compact connected Riemann surface and $U \subset N$ an open subset such that $U \cong M \backslash \{p\}$ as Riemann surfaces. Then this isomorphism extends to $N \cong M$.*

Stated more informally, there is no difference between the 'punctured Riemann surface' $M \backslash \{p\}$ and the 'marked Riemann surface' (M, p). In fact, although we have stated the theorem in terms of *compact* Riemann surfaces, the result itself is local:



(1)

In this picture the punctured open disc is assumed to be conformally isomorphic to the open set U (but nothing is supposed concerning the boundary ∂U of U in N). We may conclude that N must be, in fact, be the disc and $U \hookrightarrow N$ the punctured disc, tautologically included. For simplicity, however, the results in this article will be formulated for compact manifolds, their local counterparts being left to the reader.

By a *conformal* manifold we shall mean a smooth manifold equipped with an equivalence class of Riemannian metrics $[g_{ab}]$ where the notion of equivalence is that $\hat{g}_{ab} = \Omega^2 g_{ab}$ for some positive smooth function Ω.

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Theorem 2. *Suppose M is a compact connected conformal manifold and $p \in M$. Suppose N is a compact connected conformal manifold and $U \subset N$ an open subset such that $U \cong M \setminus \{p\}$ as conformal manifolds. Then this isomorphism extends to $N \cong M$.*

Since an oriented conformal structure in 2 dimensions is the same as a complex structure, Theorem 2 generalises Theorem 1. It is well known, however, that conformal geometry in dimensions ≥ 3 enjoys a greater rigidity than in 2 dimensions and so one expects a different proof. Such proofs of Theorem 2 (and beyond) can be found in [1]. In this article, however, we shall prove Theorem 2 by a method that also works (but much more easily so) in dimension 2.

Remark. In fact, we learned from Ben Warhurst that our proof in 2 dimensions closely follows the notion and calculation of *extremal length*, due to Ahlfors and Beurling, who show [2, Lemma 5] that the extremal length of the family of curves winding once around a 2-dimensional puncture is zero, whereas the same family viewed on $N \not\cong M$ would have strictly positive extremal length. The difficulty in extending this reasoning to higher dimensions lies in our choice of curves used to 'trap' the puncture. In Lemma 1 our choice of curves is itself based on the conformal factor and this idea seems to be new. Other than this, our proof is elementary, using only the Hölder inequality in higher dimensions as a replacement for Cauchy–Schwarz in two dimensions. Warhurst has constructed a proof in higher dimensions using, instead, the family of curves passing through the puncture. The key to his alternative proof is the fact that compact Riemannian n-manifolds are 'n-Loewner,' see for example [3].

2. Puncture repair in 2 dimensions

Proof of Theorem 1. With reference to picture (1), introducing polar coördinates (r, θ) on the disc and hence on U, we are confronted by a smooth positive function $\Omega(r, \theta)$ so that, if the η_{ab} denotes the standard metric $dr^2 + r^2 d\theta^2$ on the disc, then the metric $\hat{\eta}_{ab} = \Omega^2 \eta_{ab}$ extends to N. If $\partial U \subset N$ contains two or more points, then the concentric curves $\{r = \epsilon\}$ as $\epsilon \downarrow 0$ have length bounded away from zero in the metric $\hat{\eta}_{ab}$. In other words, for some $\epsilon > 0$ and $\ell > 0$, we have

$$\int_0^{2\pi} \Omega(r, \theta)\, r\, d\theta \geq \ell, \qquad \forall\, 0 < r < \epsilon.$$

By the Cauchy–Schwarz inequality, for any fixed r,

$$\left(\int_0^{2\pi} \Omega(r, \theta)\, d\theta\right)^2 \leq 2\pi \int_0^{2\pi} \Omega^2(r, \theta)\, d\theta$$

and it follows that

$$\int_0^{\epsilon}\int_0^{2\pi} \Omega^2\, d\theta\, r\, dr \geq \int_0^{\epsilon} \frac{1}{2\pi}\left(\int_0^{2\pi} \Omega\, d\theta\right)^2 r\, dr \geq \frac{1}{2\pi}\int_0^{\epsilon} \frac{\ell^2}{r}\, dr = \infty.$$

However, the integral on the left is the area of $\{0 < r < \epsilon\} \subseteq U$ in the metric $\hat{\eta}_{ab}$, which must be finite if $\hat{\eta}_{ab}$ is to extend smoothly to N. $\square$

3. Puncture repair in Euclidean *n*-space

In 2 dimensions, the local existence of isothermal coördinates implies that it is sufficient to repair only the unit disc in $\mathbb{R}^2$ with its standard metric η_{ab}. Such a normalisation is unavailable in higher dimensions.

Proof of Theorem 2 in flat space. With reference to (1), now viewed as a picture in n dimensions, we shall suppose that the object on the left is a punctured ball in $\mathbb{R}^n$ with its standard Euclidean metric and aim to conclude, just as we did in case $n = 2$, that $\partial U \subset N$ is a single point. To do this, we replace polar coördinates by spherical coördinates

$$\mathbb{R}_{>0} \times \Sigma \ni (r, x) \mapsto rx \in \mathbb{R}^n \setminus \{0\},$$

where

$$\Sigma = \{x \in \mathbb{R}^n \mid \|x\| = 1\} \hookrightarrow \mathbb{R}^n \setminus \{0\}$$

is the unit $(n-1)$-sphere and investigate the behaviour of a smooth positive function $\Omega = \Omega(r, x)$ defined for r sufficiently small and having the property that the metric $\hat{\eta}_{ab} = \Omega^2 \eta_{ab}$ extends to N. If $\partial U \subset N$ contains two or more points, then the concentric hypersurfaces $\{r = \epsilon\}$ as $\epsilon \downarrow 0$ have diameter bounded away from zero in the metric $\hat{\eta}_{ab}$. In other words, for some $\epsilon > 0$ and $\ell > 0$, we have

$$\forall\, 0 < r < \epsilon, \quad \text{there are } \alpha, \beta \in \Sigma \quad \text{s.t.} \int_\alpha^\beta \Omega(r, x)\, r \geq \ell,$$

where the integral is along any path from α to β on the unit sphere Σ (with respect to the standard round metric on Σ).