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Prescribing the mixed scalar curvature of a foliated Riemann–Cartan manifold

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ABSTRACT

The mixed scalar curvature is the simplest curvature invariant of a foliated Riemannian manifold. We explore the problem of prescribing the leafwise constant mixed scalar curvature of a foliated Riemann–Cartan manifold by conformal change of the structure in tangent and normal to the leaves directions. Under certain geometrical assumptions and in two special cases: along a compact leaf and for a closed fibered manifold, we reduce the problem to solution of a nonlinear leafwise elliptic equation for the conformal factor. We are looking for its solutions that are stable stationary solutions of the associated parabolic equation. Our main tool is using of majorizing and minorizing nonlinear heat equations with constant coefficients and application of comparison theorems for solutions of Cauchy's problem for parabolic equations.

0. Introduction

Geometrical problems of prescribing curvature of a Riemannian manifold (M, g) using conformal change of metric g are popular for a long time, for example, the study of constancy of the scalar curvature was began by Yamabe in 1960 and completed by Trudinger, Aubin and Schoen in 1986, see [1]. This geometrical problem is expressed in terms of the existence and multiplicity of solutions of a given elliptic PDE in the Riemannian manifold. Several authors developed an analogue of the problem for CR-manifolds and contact manifolds.

Foliations arise in topology, geometry and analysis; many models in physics are foliated. The simplest curvature invariant of a foliation $\mathcal{F}$ (that is partition of a manifold into collections of submanifolds called leaves) is the *mixed scalar curvature*, $\mathrm{S}_{\mathrm{mix}}$, see [2,3]. This is an averaged sectional curvature over all *mixed planes* — planes containing vectors from both distributions: tangent to leaves, $\mathcal{D} = T\mathcal{F}$, and normal, $\mathcal{D}^\perp = (T\mathcal{F})^\perp$. The problem of prescribing leafwise constant $\mathrm{S}_{\mathrm{mix}}$ was studied by authors in [4–6], using flows of $\mathcal{D}^\perp$-conformal metrics.

The metrically-affine geometry, founded by E. Cartan in 1923–1925, generalizes Riemannian geometry and uses an asymmetric connection $\bar\nabla$ instead of the Levi-Civita connection ∇ of g; in extended theory of gravity the torsion of $\bar\nabla$ is represented by the spin tensor of matter. Notice that $\bar\nabla$ and ∇ are projectively equivalent (have the same geodesics) if and only if the *contorsion (difference) tensor* $\mathfrak{T} := \bar\nabla - \nabla$ is antisymmetric. *Riemann–Cartan* (RC) spaces have metric connection, $\bar\nabla g = 0$, and appear in such topics as homogeneous and almost Hermitian spaces [7], and flows of metrics [8].

Let $\{E_a,\ \mathcal{E}_i\}_{a\le p,\, i\le n}$ be a local orthonormal frame on TM such that $\{E_a\} \subset \mathcal{D}$ and $\{\mathcal{E}_i\} \subset \mathcal{D}^\perp$ and $\epsilon_a = g(E_a, E_a),\ \epsilon_i = g(\mathcal{E}_i, \mathcal{E}_i)$. We use the convention for various tensors: $\mathfrak{T}_i = \mathfrak{T}_{\mathcal{E}_i}$, etc.

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Definition 1. The *mixed scalar curvature* of the curvature tensor $\bar{R}$ of $\bar{\nabla}$ on a foliated metric-affine (in particular, RC) manifold $(M, g, \bar{\nabla})$ is the following function:

$$\bar{\mathrm{S}}_{\mathrm{mix}} = \frac{1}{2}\sum_{a,i}\epsilon_a\epsilon_i\big(g(\bar{R}_{E_a,\mathcal{E}_i}E_a,\,\mathcal{E}_i) + g(\bar{R}_{\mathcal{E}_i,E_a}\mathcal{E}_i,\,E_a)\big). \tag{1}$$

The definition (1) does not depend on the order of distributions and on the choice of a local frame, and generalizes the notion of the *mixed scalar curvature* of the curvature tensor R of ∇ on a foliated pseudo-Riemannian manifold (M, g), see [2,3]:

$$\mathrm{S}_{\mathrm{mix}} = \sum_{a,i}\epsilon_a\epsilon_i\, g(R_{E_a,\mathcal{E}_i}E_a,\,\mathcal{E}_i). \tag{2}$$

In the paper, we study the **problem** of prescribing the leafwise constant $\bar{\mathrm{S}}_{\mathrm{mix}}$ of a foliated RC manifold by conformal change of RC structure in $\mathcal{D}$ and $\mathcal{D}^\perp$ directions:

(**P**). *Given foliated RC manifold* $(M, g, \bar{\nabla} = \nabla + \mathfrak{T})$ *find smooth functions on* M: $u > 0$ *and a leafwise constant* Φ *such that the* $(\mathcal{D}, \mathcal{D}^\perp)$*-conformal RC structure, that is*

$$g' = g^\top \oplus u^2\, g^\perp, \quad \mathfrak{T}' = u^2\,\mathfrak{T}^\top \oplus \mathfrak{T}^\perp, \tag{3}$$

has the mixed scalar curvature $\bar{\mathrm{S}}_{\mathrm{mix}} = \Phi$.

Here $X^\top$ is the $\mathcal{D}$-component of $X \in TM$ (resp., $X^\perp$ is the $\mathcal{D}^\perp$-component of X), and

$$\begin{aligned} &g^\top(X, Y) := g(X^\top, Y^\top), \quad g^\perp(X, Y) := g(X^\perp, Y^\perp),\\ &\mathfrak{T}^\top_X Y := (\mathfrak{T}_X Y)^\top, \qquad \mathfrak{T}^\perp_X Y := (\mathfrak{T}_X Y)^\perp. \end{aligned} \tag{4}$$

The problem (**P**) seems to be interesting since the geometry of foliated RC manifolds has not been studied at all. We observe (see Section 1.2) that under some geometric assumptions, including $g_{|\mathcal{D}} > 0$, the conformal factor in (3) obeys a leafwise elliptic equation. Since the topology of the leaf through a point can change crucially with the point, it is difficult to study leafwise elliptic PDEs. Thus, we examine two formulations of the problem:

Find a pair $(u,\ \Phi)$ *in* problem (**P**) *to prescribe*:

(i) *constant* $\bar{\mathrm{S}}_{\mathrm{mix}}$ *on a compact leaf* F.

(ii) *smooth leafwise constant* $\bar{\mathrm{S}}_{\mathrm{mix}}$ *on a closed* M, *fibered instead of being foliated*:

$$\mathcal{F} \text{ is defined by an orientable fiber bundle } \pi : M \to B. \tag{5}$$

Remark 1. The existence of compact leaves (for example, Novikov's theorem for M^3 foliated by surfaces) and their stability are important problems in foliations theory. Integral formulas provide obstructions for existence of foliations or compact leaves of them with given geometric properties see surveys in [2,9]; such formulas for foliated RC manifolds were obtained in [10].

The $\mathcal{K}$-sectional curvature of any symmetric $(1, 2)$-tensor $\mathcal{K}$ has been defined in [11] and studied for statistical manifolds. In the work we introduce apply the following algebraic analogue of $\mathrm{S}_{\mathrm{mix}}$ for any $(1, 2)$-tensor $\mathcal{K}$ on a foliation.

Definition 2. The *mixed scalar* $\mathcal{K}$*-curvature* of arbitrary $(1, 2)$-tensor $\mathcal{K}$ on a foliated manifold is an averaged $\mathcal{K}$-sectional curvature over all mixed planes:

$$\mathrm{S}_{\mathcal{K}} := \frac{1}{2}\sum_{a,i}\epsilon_a\epsilon_i\big(g(\,[\mathcal{K}_i,\ \mathcal{K}_a]\,E_a,\,\mathcal{E}_i) + g(\,[\mathcal{K}_a,\ \mathcal{K}_i]\,\mathcal{E}_i,\,E_a)\big). \tag{6}$$

If $\mathcal{K}_X$ $(X \in TM)$ is (anti-)symmetric then (6) reads $\mathrm{S}_{\mathcal{K}} = \sum_{a,i}\epsilon_a\epsilon_i\, g(\,[\mathcal{K}_i,\ \mathcal{K}_a]\,E_a,\,\mathcal{E}_i)$.

Since *RC spaces* have metric compatible connection, $\mathfrak{T}_X$ $(X \in TM)$ is anti-symmetric:

$$(\bar{\nabla}_X\, g)(Y, Z) = g(\mathfrak{T}_X Y, Z) + g(\mathfrak{T}_X Z, Y) = 0 \quad (X, Y, Z \in TM). \tag{7}$$

Thus, see (6), the mixed scalar $\mathfrak{T}$-curvature (that is for $\mathcal{K} = \mathfrak{T}$) in RC case is

$$\mathrm{S}_{\mathfrak{T}} := \sum_{a,i}\epsilon_a\epsilon_i\, g(\,[\mathfrak{T}_a,\ \mathfrak{T}_i]\,\mathcal{E}_i,\,E_a). \tag{8}$$

Since $\mathfrak{T}^\top$, see (4), obeys $g(\,[\mathfrak{T}^\top_i,\ \mathfrak{T}^\top_a]\,E_a,\,\mathcal{E}_i) = 0$, by (6) with $\mathcal{K} = \mathfrak{T}^\top$ and any $\bar{\nabla}$ we get

$$\mathrm{S}_{\mathfrak{T}^\top} := \sum_{a,i}\epsilon_a\epsilon_i\, g(\,[\mathfrak{T}^\top_a,\ \mathfrak{T}^\top_i]\,\mathcal{E}_i,\,E_a). \tag{9}$$