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Harmonic maps of finite energy for Finsler manifolds

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ABSTRACT

In this paper, we study some properties of harmonic maps for Finsler manifolds. Some Liouville theorems on harmonic maps for Finsler manifolds are given. Let M be a complete simply connected Riemannian manifold with non-negative Ricci curvature and $\overline{M}$ be a complete Berwald manifold with non-positive flag curvature. The main purpose of this paper is to prove that there exists no non-degenerate harmonic map ϕ from M to $\overline{M}$ with $\int_{SM} e(\phi) dV_{SM} < \infty$, which generalizes the result of Schoen and Yau (1976) from Riemannian manifolds to Berwald manifolds.

1. Introduction

Harmonic maps between Riemannian manifolds are important in both classical and modern differential geometry. There are many studies on harmonic maps between Riemannian manifolds. The following theorem is well-known:

Theorem A (*[1]*). *Any harmonic map ϕ from a compact Riemannian manifold M with nonnegative Ricci curvature to a Riemannian manifold with non-positive section curvature must be totally geodesic.*

A Finsler manifold is a differentiable manifold with Finsler metric. A Finsler metric is just a Riemannian metric without the quadratic restriction. Recent studies on Finsler manifolds have taken on a new look and Finsler manifolds can also be applied to biology, physics, etc. A harmonic map between Finsler manifolds is defined as the critical point of its energy functional by means of the volume measure induced from the projective sphere bundle of the Finsler manifold. Several scholars have made fundamental contributions to this subject [2–10]. In 2007, He and Shen considered harmonic maps from a compact Riemannian manifold to a Berwald manifold, and by using the Bochner-type formula for non-degenerate map between Finsler manifolds given in [3], they proved the following:

Theorem B (*[2]*). *Any non-degenerate harmonic map from a compact Riemannian manifold with nonnegative sectional curvature to a Berwald manifold with non-positive flag curvature must be totally geodesic.*

He and Zheng improve the above result of He and Shen from the assumption condition nonnegative sectional curvature to the assumption condition nonnegative Ricci curvature, i.e. they obtained the following:

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Theorem C ([4]). *Any non-degenerate harmonic map from a compact Riemannian manifold with non-negative Ricci curvature to a Berwald manifold with non-positive flag curvature must be totally geodesic.*

Author considered harmonic maps from a Berwald manifold to a Landsberg manifold and can obtain the following:

Theorem D ([6]). *Any non-degenerate strongly harmonic map from a compact Berwald manifold with nonnegative general Ricci curvature to a Landsberg manifold with non-positive flag curvature must be totally geodesic.*

When the source manifold M is a complete Riemannian manifold, by using the fact that every complete non-compact Riemannian manifold with non-negative Ricci curvature has infinite volume, Schoen and Yau proved the following theorem:

Theorem E ([8]). *Let M be a complete Riemannian manifold with non-negative Ricci curvature and $\overline{M}$ be a complete Riemannian manifold with non-positive sectional curvature. If $\phi : M \to \overline{M}$ is a harmonic map with $\int_M e(\phi)dv < \infty$, then ϕ is constant.*

But when the source manifold M is a complete Finsler manifold. Since the energy density $e(\phi)$ of a harmonic map ϕ depends on not only the point $x \in M$ but also $Y \in T_xM$, it would be difficult to prove that the energy density $e(\phi)$ of a harmonic map ϕ is independent of Y, the Liouville theorems of harmonic maps for complete Finsler manifolds is still open. The main purpose of this paper is to extend the above Theorem E to Berwald manifolds. In this paper, by means of the properties of the Euclidean space, we shall prove the following:

Main Theorem. Let M be a complete simply connected Riemannian manifold with non-negative Ricci curvature and $(\overline{M}, \overline{F})$ be a complete Berwald manifold with non-positive flag curvature. If

$$\lim_{R\to\infty} \frac{\int_{SB_R(x_0)} e(\phi) * 1}{R} = 0,$$

where $SB_R(x_0) = \cup_{x\in B_R(x_0)} S_xM$ and $B_R(x_0)$ is a geodesic ball of M with radius R centered at a fixed point $x_0 \in M$, then there exists no non-degenerate harmonic map $\phi : M \to \overline{M}$.

Corollary 3.1 ([11]). *Let M be a complete simply connected Riemannian manifold with non-negative Ricci curvature and $\overline{M}$ be a complete Riemannian manifold with non-positive sectional curvature. If*

$$\lim_{R\to\infty} \frac{\int_{B_R(x_0)} e(\phi) * 1}{R} = 0,$$

then any harmonic map $\phi : M \to \overline{M}$ is constant.

When the source manifold is a Finsler manifold (M, F), Qun and Shen proved the following theorem:

Theorem F ([2]). *A strongly harmonic map ϕ from a compact Finsler manifold with nonnegative flag curvature to a Riemannian manifold with non-positive sectional curvature must be totally geodesic.*

In this paper, we consider the strongly harmonic map ϕ from a complete Finsler manifold to a Landsberg manifold and also have the following:

Theorem 4.2. *Let (M, F) be a complete Finsler manifold with non-negative flag curvature and $\overline{M}$ be a complete Landsberg manifold with non-positive flag curvature. If*

$$\lim_{R\to\infty} \frac{\int_{SB_R(x_0)} e(\phi) * 1}{R} = 0,$$

then any strongly harmonic map $\phi : M \to \overline{M}$ must be totally geodesic. In particular, if the flag curvature of M is strictly positive definite at some point, then there exists no non-degenerate strongly harmonic map $\phi : (M, F) \to (\overline{M}, \overline{F})$.

2. Preliminaries

We shall use the following convention of index ranges unless otherwise stated:

$$1 \le i, j, \ldots \le n; \quad 1 \le \alpha, \beta, \ldots \le m; \quad 1 \le a, b, \ldots \le n-1.$$

Let M be an n-dimensional smooth manifold and $\pi : TM \to M$ be the natural projection from the tangent bundle. Let (x, Y) be a point of TM with $x \in M$, $Y \in T_xM$ and let (x^i, Y^i) be the local coordinates on TM with $Y = Y^i \frac{\partial}{\partial x^i}$. A Finsler metric on M is a function $F : TM \to [0, +\infty)$ satisfying the following properties:

(i) Regularity: $F(x, Y)$ is smooth in $TM \setminus 0$;
(ii) Positive homogeneity: $F(x, \lambda Y) = \lambda F(x, Y)$ for $\lambda > 0$;
(iii) Strong convexity: the fundamental quadratic form $g = g_{ij}dx^i \otimes dx^j$ is positive definite, where $g_{ij} = \frac{\partial^2(F^2)}{2\partial Y^i \partial Y^j}$.