Journal of Geometry and Physics 126 (2018) 7–21

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys

Symmetric products of a real curve and the moduli space of Higgs bundles

Thomas John Baird

Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, NL, Canada, A1C 5S7

ARTICLE INFO

Article history:
Received 28 April 2017
Received in revised form 22 December 2017
Accepted 3 January 2018
Available online 11 January 2018

Keywords:
Moduli spaces of Higgs bundles
Symmetric products of a curve
Spectral sequences
Betti numbers
Anti-holomorphic involutions

ABSTRACT

Consider a Riemann surface X of genus $g \geq 2$ equipped with an antiholomorphic involution τ. This induces a natural involution on the moduli space $M(r,d)$ of semistable Higgs bundles of rank r and degree d. If D is a divisor such that $\tau(D) = D$, this restricts to an involution on the moduli space $M(r,D)$ of those Higgs bundles with fixed determinant $\mathcal{O}(D)$ and trace-free Higgs field. The fixed point sets of these involutions $M(r,d)^\tau$ and $M(r,D)^\tau$ are (A,A,B)-branes introduced by Baraglia and Schaposnik (2016). In this paper, we derive formulas for the mod 2 Betti numbers of $M(r,d)^\tau$ and $M(r,D)^\tau$ when $r = 2$ and d is odd. In the course of this calculation, we also compute the mod 2 cohomology ring of $Sym^m(X)^\tau$, the fixed point set of the involution induced by τ on symmetric products of the Riemann surface.

1. Introduction

Let X denote a compact, connected Riemann surface with canonical bundle K. Given a complex vector bundle E over X, the rank $rank(E)$ is the dimension of a fibre and the degree $deg(E) := c_1(E)(X)$ is the integral of the first Chern class. A *Higgs bundle* (E, Φ) over X consists of a holomorphic vector bundle E over X and a section $\Phi \in H^0(X, Hom(E, E \otimes K))$ called the *Higgs field*. A Higgs field is called *stable* if all proper vector subbundles $F \subset E$ such that $\Phi(F) \subseteq F \otimes K$ satisfy $deg(F)/rank(F) \leq deg(E)/rank(E)$. In [1], Hitchin constructed the moduli space $M(r,d)$ of semistable Higgs bundles of rank r and degree d. We will always assume that r and d are coprime and X has genus $g \geq 2$, so $M(r,d)$ is non-singular.

Fix a divisor $D \in Div(X)$. Define $M(r,D)$ to be the subvariety of $M(r,d)$ of Higgs bundles (E,Φ) for which $\wedge^r E \cong \mathcal{O}(D)$ and $tr(\Phi) = 0$. Both $M(r,d)$ and $M(r,D)$ admit a complete hyperkähler metric: a Riemannian metric g which is Kähler with respect to three different complex structures I, J, K that satisfy the quaternionic relations. We denote by $\omega_I, \omega_J, \omega_K$ the corresponding Kähler forms.

Suppose that X admits an anti-holomorphic involution τ and call (X,τ) a *real curve*. This induces an involution on $M(r,d)$ (which we also call τ) sending a pair (E,Φ) to $\tau(E,\Phi) := (\tau(E), \tau(\Phi))$, where $\tau(E) = \tau^*\overline{E}$ is the conjugate pull-back and $\tau(\Phi)$ is the composition

$$\tau^*\overline{E} \xrightarrow{(\tau^*)^{-1}} E \xrightarrow{\Phi} E \otimes K \xrightarrow{\tau^*} \tau^*(\overline{E} \otimes \overline{K}) \xrightarrow{\cong} \tau^*(\overline{E}) \otimes K$$

where we have used the natural isomorphism $K \cong \tau^*\overline{K}$ determined by the fact that τ is anti-holomorphic. If $D \in Div(X)$ is a *real divisor* in the sense that $\tau(D) = D$, then τ restricts to an involution of $M(r,D)$.

This involution was considered by Baraglia–Schaposnik [2] (they denote it i_3). It preserves the hyperkähler metric, is anti-holomorphic with respect to I, J and is holomorphic with respect to K. Consequently, the fixed point sets of the involutions $M(r,d)^\tau$ and $M(r,D)^\tau$ are real and Lagrangian with respect to I, J and are complex and symplectic with respect to K. Such

E-mail address: tbaird@mun.ca.

https://doi.org/10.1016/j.geomphys.2018.01.003
0393-0440/

a submanifold is called an (A, A, B)-brane, which play a role in the Kapustin–Witten approach to geometric Langlands duality [3–5] and this duality was explored for $M(r, d)^\tau$ and $M(r, D)^\tau$ in [2]. In the current paper, we derive formulas computing the mod 2 Betti numbers of $M(r, d)^\tau$ and $M(r, D)^\tau$ in the case when the rank $r = 2$ and the degree d is odd.

1.1. Outline of the proof

There is natural $\mathbb{C}^*$-action on $M(r, D)$ by scaling the Higgs field. Hitchin observed that the restricted $U(1)$-action is Hamiltonian with respect to the symplectic structure ω_I, with proper moment map $\mu : M(r, D) \to \mathbb{R}$,

$$\mu(E, \Phi) = \|\Phi\|^2_{L^2}\,.$$

Therefore, by a theorem of Frankel [6], the function μ is a perfect Morse–Bott function with respect to rational coefficients and the critical points of μ coincide with the $U(1)$-fixed points. This means we have an equality

$$P_t^{\mathbb{Q}}(M(r, D)) = \sum_{F \text{ component of } M(r,D)^{U(1)}} P_t^{\mathbb{Q}}(F) t^{2d_F}$$

where $P_t^{\mathbb{Q}}(Y) := \sum_{i=0}^{\infty} \dim(H^i(Y; \mathbb{Q}))t^i$ is the rational Poincaré series and $2d_F$ is the Morse index of the path component F (which is necessarily even because the negative normal bundles are symplectic). This reduces the calculation of the rational Betti numbers of $M(r, D)$ to calculating the Betti numbers of the fixed point components F and their Morse indices $2d_F$. This was carried out for rank $r = 2$ by Hitchin [1], for rank $r = 3$ by Gothen [7], and rank $r = 4$ by García-Prada, Heinloth, and Schmitt [8].

Similar considerations apply to compute mod 2 Betti numbers of $M(r, D)^\tau$. The involution is compatible with the $U(1)$-action in the sense that $e^{i\theta} \circ \tau = \tau \circ e^{-i\theta}$ and $\mu \circ \tau = \mu$. In this circumstance, a theorem of Duistermaat [9,10] tells us that the restriction of μ to $M(r, D)^\tau$ is a perfect Morse–Bott function with respect to mod 2 coefficients. The set of critical points of μ restricted to $M(r, D)^\tau$ coincides with $M(r, D)^\tau \cap M(r, D)^{U(1)}$ and the Morse indices are halved (since they compute the dimension of Lagrangian vector subbundles of symplectic vector bundles). Consequently, we obtain the formula

$$P_t(M(r, D)^\tau) = \sum_{F \text{ component of } M(r,D)^{U(1)}} P_t(F^\tau) t^{d_F} \tag{1.1}$$

where $P_t(Y) := \sum_{i=0}^{\infty} \dim(H^i(Y; \mathbb{Z}_2))t^i$ is the $\mathbb{Z}_2$-Poincaré series. Thus to compute the mod 2 Betti numbers of $M(r, D)^\tau$ it remains only to compute those of F^τ.

One path component of $M(r, D)^{U(1)}$ coincides with the global minimum of μ. Since μ is minimized on $M(r, D)$ exactly when the Higgs field vanishes, the minimizing set is identified with the *moduli space of stable vector bundles* $N(r, D)$ of rank r and determinant $\mathcal{O}(D)$. The global minimizing set of μ restricted to $M(r, D)^\tau$ is consequently identified with $N(r, D)^\tau$, the *moduli space of real vector bundles* of rank r and determinant $\mathcal{O}(D)$. This moduli space was introduced in [11,12] and its mod 2 Betti numbers were computed in [13–15] for all coprime ranks and degrees.

We restrict now to the case where the rank $r = 2$, where the higher strata admit a simple description (we hope to consider the higher rank case in future). When $r = 2$, Hitchin shows that the remaining $U(1)$-fixed points are represented by pairs (E, Φ) of the form

$$E = L \oplus (L^* \otimes \mathcal{O}(D)), \quad \Phi = \begin{bmatrix} 0 & 0 \\ \varphi & 0 \end{bmatrix}$$

where $\varphi \in H^0(L^{-2} \otimes K(D))$. The fixed point components are identified with pullbacks F_l of the form

$$\begin{array}{ccc} F_l & \longrightarrow & Pic_l(X) \\ \downarrow & & \downarrow{\scriptstyle sq} \\ Sym^m(X) & \xrightarrow{aj} & Pic_m(X) \end{array} \tag{1.2}$$

where $Sym^m(X)$ is the m-fold symmetric product of X, aj is the Abel–Jacobi map, sq is the map sending $[L]$ to $[L^{-2} \otimes K(D)]$, $m = 2g - 2 - 2l + d$, and l ranges between $1 \leq l \leq g - 1$. Here, sq is a 2^{2g}-fold covering map that can be identified with the squaring map under an appropriate translation $Pic_l(X) \cong Pic_m(X)$ to the Jacobian $Jac(X) := Pic_0(X)$ which is isomorphic to $U(1)^{2g}$ as a Lie group.

The diagram (1.2) is equivariant with respect to the induced τ-actions and we identify F_l^τ with the pull-back of the restriction to τ-fixed points

$$\begin{array}{ccc} F_l^\tau & \longrightarrow & Pic_l(X)^\tau \\ \downarrow & & \downarrow{\scriptstyle sq} \\ Sym^m(X)^\tau & \xrightarrow{aj} & Pic_m(X)^\tau\,. \end{array} \tag{1.3}$$