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Varieties of sums of powers and moduli spaces of $(1, 7)$-polarized abelian surfaces

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ABSTRACT

We study the geometry of some varieties of sums of powers related to the Klein quartic. This allows us to describe the birational geometry of certain moduli spaces of abelian surfaces. In particular we show that the moduli space $\mathcal{A}_2(1,7)^-_{sym}$ of $(1, 7)$-polarized abelian surfaces with a symmetric theta structure and an odd theta characteristic is unirational by showing that it admits a dominant morphism from a unirational conic bundle.

1. Introduction

Varieties of sums of powers, VSP for short, parametrize decompositions of a general homogeneous polynomial $F \in k[x_0, \dots, x_n]_d$ as sums of powers of linear forms. They have been widely studied from both the biregular [1–3] and the birational viewpoint [4,5], and furthermore in relation to secant varieties [6–9].

The relation of $\mathcal{A}_2(1,7)$, the moduli space of abelian surfaces with a polarization of type $(1, 7)$ and a $(1, 7)$-level structure, with the variety of sums of powers of the Klein quartic dates back to the work of S. Mukai [2].

In this paper we investigate the birational geometry of some moduli spaces of abelian surfaces related to $\mathcal{A}_2(1,7)$. In particular, if we endow the abelian surfaces in $\mathcal{A}_2(1,7)$ with a symmetric theta structure and a theta characteristic (odd or even), we obtain two new moduli spaces, $\mathcal{A}_2(1,7)^-_{sym}$ and $\mathcal{A}_2(1,7)^+_{sym}$, that are finite covers of degree 6 and 10 respectively of $\mathcal{A}_2(1,7)$. For a general introduction to these spaces see [10, Sections 6.1.1]. We introduce also the moduli space $\mathcal{A}_2(1,7;2,2)$ parametrizing abelian surfaces with a polarization of type $(1, 7)$, a $(1, 7)$-level structure and a $(2, 2)$-level structure.

Furthermore, we introduce new types of varieties of sums of powers and showcase rational maps between them and our moduli spaces of abelian surfaces. The first variety of sums of powers we take into account is a variety where we also allow

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ourselves to fix an order on the linear forms that make up the decomposition of the polynomial. Let $\nu_d^n : \mathbb{P}^n \to \mathbb{P}^{N(n,d)}$, with $N(n, d) = \binom{n+d}{d} - 1$ be the degree d Veronese embedding of $\mathbb{P}^n$, and let $V_d^n = \nu_d^n(\mathbb{P}^n)$ be the corresponding Veronese variety.

Definition 1.1. Let F be a polynomial of degree d in $n + 1$ variables. We define

$$\mathrm{VSP}_{ord}(F, h)^o := \{(L_1, \dots, L_h) \in (\mathbb{P}^{n*})^h \mid F \in \langle L_1^d, \dots, L_h^d \rangle \subseteq \mathbb{P}^{N(n,d)}\} \subseteq (\mathbb{P}^{n*})^h$$

and $\mathrm{VSP}_{ord}(F, h) := \overline{\mathrm{VSP}_{ord}(F, h)^o}$ by taking the closure of $\mathrm{VSP}_{ord}(F, h)^o$ in $(\mathbb{P}^{n*})^h$.

Then we consider the natural action of the symmetric group S_h on $\mathrm{VSP}_{ord}(F, h)$ and two related variations on the classical definition of VSP.

Definition 1.2. Consider the action of S_{h-1} on $\mathrm{VSP}_{ord}(F, h)$ given by permuting the linear forms $(L_2, \dots, L_h)$. The variety $\mathrm{VSP}_h(F, h)$ is the quotient

$$\mathrm{VSP}_h(F, h) = \mathrm{VSP}_{ord}(F, h)/S_{h-1}.$$

If $h = 2r$ is even, we consider the action of $S_r \times S_r$ on $\mathrm{VSP}_{ord}(F, h)$ such that the first and the second copy of S_r act on the first and the last r linear forms respectively. Let $X_h(F) = \mathrm{VSP}_{ord}(F, h)/S_r \times S_r$. This space comes with a natural S_2-action switching $\{L_1, \dots, L_r\}$ and $\{L_{r+1}, \dots, L_h\}$. We define

$$\mathrm{VSP}^h(F, h) := X_h(F)/S_2.$$

When $F = F_4$ is the Klein quartic we will denote $\mathrm{VSP}^6(F, 6)$ by $\mathrm{VSP}_{10}(F_4, 6)$ and $\mathrm{VSP}_{ord}(F, 6)$ by $\mathrm{VSP}_{720}(F_4, 6)$. The first main result of this paper is the following.

Theorem 1.3. *The moduli spaces $\mathcal{A}_2(1, 7)^-_{sym}$ and $\mathcal{A}_2(1, 7)^+_{sym}$ are birational to the varieties $\mathrm{VSP}_6(F_4, 6)$ and $\mathrm{VSP}_{10}(F_4, 6)$ respectively, where $F_4 \in k[x_0, x_1, x_2]_4$ is the Klein quartic. Furthermore, the moduli space $\mathcal{A}_2(1, 7; 2, 2)$ is birational to $\mathrm{VSP}_{720}(F_4, 6)$.*

We summarize the situation in the following diagram, where the superscripts on the arrows indicate the degrees of the respective maps.

$$\begin{tikzcd}
 & \mathrm{VSP}_{720}(F_4, 6) \overset{\mathrm{bir}}{\cong} \mathcal{A}_2(1, 7; 2, 2) \arrow[dl, "3!"'] \arrow[d, "4!"] \arrow[dr, dashed, "5!"] & \\
\mathcal{P}_3 \arrow[r, "4"] \arrow[d, dashed, "12"'] \arrow[drr, dashed, "120"] & \mathcal{P}_2 \arrow[r, dashed, "5"] \arrow[dr, dashed, "30"] & \mathrm{VSP}_6(F_4, 6) \overset{\mathrm{bir}}{\cong} \mathcal{A}_2(1, 7)^-_{sym} \arrow[d, dashed, "6"] \\
\mathrm{VSP}_{10}(F_4, 6) \overset{\mathrm{bir}}{\cong} \mathcal{A}_2(1, 7)^+_{sym} \arrow[rr, dashed, "10"] & & \mathrm{VSP}(F_4, 6) \overset{\mathrm{bir}}{\cong} \mathcal{A}_2(1, 7)
\end{tikzcd}$$

The apolarity theory developed in [11] allows us to produce in Section 2 a 3-fold conic bundle dominating $\mathrm{VSP}_6(F_4, 6)$, and to conclude that it is unirational. In Section 4 we develop some birational geometry of the moduli spaces of abelian surfaces that we are considering. In particular, as a consequence of the above result, we get the following.

Theorem 1.4. *The moduli space $\mathcal{A}_2(1, 7)^-_{sym}$ is unirational.*

At the end of the paper we also propose some open questions on the birational geometry of $\mathcal{A}_2(1, 7)^+_{sym}$ and $\mathcal{A}_2(1, 7; 2, 2)$. Throughout all the paper we will work over the complex field.

2. Ordered varieties of sums of powers

Let $\nu_d^n : \mathbb{P}^n \to \mathbb{P}^{N(n,d)}$, with $N(n, d) = \binom{n+d}{d} - 1$ be the Veronese embedding induced by $\mathcal{O}_{\mathbb{P}^n}(d)$, and let $V_d^n = \nu_d^n(\mathbb{P}^n)$ be the corresponding Veronese variety.

Definition 2.1. Let $F \in k[x_0, \dots, x_n]_d$ be a general homogeneous polynomial of degree d. Let h be a positive integer and $\mathrm{Hilb}_h(\mathbb{P}^{n*})$ the Hilbert scheme of sets of h points in $\mathbb{P}^{n*}$. We define

$$\mathrm{VSP}(F, h)^o := \{\{L_1, \dots, L_h\} \in \mathrm{Hilb}_h(\mathbb{P}^{n*}) \mid F \in \langle L_1^d, \dots, L_h^d \rangle \subset \mathbb{P}^{N(n,d)}\} \subseteq \mathrm{Hilb}_h(\mathbb{P}^{n*}),$$

and $\mathrm{VSP}(F, h) := \overline{\mathrm{VSP}(F, h)^o}$ by taking the closure of $\mathrm{VSP}(F, h)^o$ in $\mathrm{Hilb}_h(\mathbb{P}^{n*})$.