Journal of Geometry and Physics 125 (2018) 49–74

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys

Minimal models of compact symplectic semitoric manifolds

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ARTICLE INFO

Article history:
Received 10 May 2017
Received in revised form 5 October 2017
Accepted 26 December 2017
Available online 5 January 2018

Keywords:
Symplectic geometry
Integrable systems
$SL_2(\mathbb{Z})$
Symplectic toric manifolds
Fans
Matrix calculus

ABSTRACT

A symplectic semitoric manifold is a symplectic 4-manifold endowed with a Hamiltonian $(S^1 \times \mathbb{R})$-action satisfying certain conditions. The goal of this paper is to construct a new symplectic invariant of symplectic semitoric manifolds, the helix, and give applications. The helix is a symplectic analogue of the fan of a nonsingular complete toric variety in algebraic geometry, that takes into account the effects of the monodromy near focus–focus singularities. We give two applications of the helix: first, we use it to give a classification of the minimal models of symplectic semitoric manifolds, where "minimal" is in the sense of not admitting any blowdowns. The second application is an extension to the compact case of a well known result of Vũ Ngọc about the constraints posed on a symplectic semitoric manifold by the existence of focus–focus singularities. The helix permits to translate a symplectic geometric problem into an algebraic problem, and the paper describes a method to solve this type of algebraic problem.

1. Introduction

The revolution in symplectic toric geometry started in the 1980s with the proof of the convexity of the image of the momentum map $F = (f_1, \dots, f_k) : (M, \omega) \to \mathbb{R}^k$ associated to a compact symplectic $2n$-manifold acted upon by a k-dimensional compact connected abelian Lie group T (i.e. a k-dimensional torus $T = (S^1)^k$), due independently to Guillemin–Sternberg [1] and Atiyah [2]. In fact, $F(M)$ is the polytope Δ equal to the convex hull of the image under F of the fixed points of the T-action. In the case that $n = k$ such manifolds are called *symplectic toric*.

Shortly after, Delzant proved [3] that in the symplectic toric case the image Δ encodes all of the information about the manifold M, the form ω, and the ω-preserving T-action. That is, Δ is the only symplectic T-equivariant invariant of (M, ω, F). He moreover showed that any simple, rational, smooth polytope Δ arises as the image of a momentum map of a symplectic-toric manifold; following Guillemin these polytopes are now called *Delzant*.

The existence of this action poses restrictions on (M, ω) and F. For instance, F only has elliptic singularities; moreover, the fibers are tori of dimension 0 up to n (in particular, they are submanifolds of M).

Delzant's classification was extended in [4,5] to compact and noncompact symplectic 4-manifolds acted upon by the noncompact Lie group $S^1 \times \mathbb{R}$, under certain assumptions (the action must be Hamiltonian, all singularities must be non-degenerate, with none of hyperbolic type, the moment map of the S^1-action must be proper, and each fiber contains at most one isolated singularity) these manifolds are called *symplectic semitoric*, and so far are classified when M is 4-dimensional. In this case the momentum map of the $(S^1 \times \mathbb{R})$-action is $F = (f_1, f_2)$, where the Hamiltonian vector field associated to f_1 is periodic, but not necessarily the one associated to f_2. The main novelty with respect to symplectic toric manifolds is that F may have, in addition to elliptic singularities, another type of singularities known as focus–focus singularities. The fiber

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https://doi.org/10.1016/j.geomphys.2017.12.005
0393-0440/

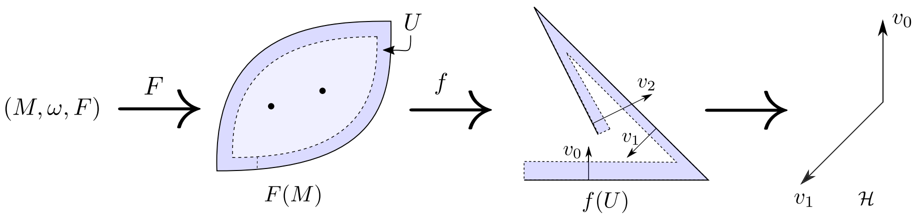

Fig. 1. The helix is intrinsically constructed by defining a toric momentum map on the preimage of U, a neighborhood of the boundary of the image of the momentum map minus a single cut, and collecting the inwards pointing normal vectors of the piecewise linear boundary of the resulting set in $\mathbb{R}^2$. Notice that v_2 does not occur in the representative of the helix we have drawn because after applying the monodromy it is equal to v_0.

containing a focus–focus singularity is not a submanifold, it is homeomorphic to a sphere with its south and north poles identified (i.e. a torus pinched at the focus–focus singularity). Symplectic semitoric manifolds are an example of almost toric manifolds, as introduced by Symington [6].

Symplectic semitoric manifolds are characterized by five invariants, one of which is a polygon P constructed from $F(M)$ according to Vũ Ngọc [7], by unfolding the singular affine structure induced by F on $F(M)$ as a subset of $\mathbb{R}^2$ (in fact $F(M)$ need not even be convex[1]). The other four invariants account for the effect of the focus–focus singularities and the monodromy around them (a fundamental phenomena studied by Duistermaat [8]). There are natural notions of blowdown in the symplectic toric and symplectic semitoric settings which we describe in Section 2.4.

Definition 1.1. A symplectic toric or symplectic semitoric manifold is *minimal* if it does not admit a blowdown.

For a symplectic toric manifold chopping off a corner of Δ corresponds to T-equivariantly blowing up M at a T-fixed point, and the inverse operation corresponds to blowing down. To Δ one can associate a *fan*, the one corresponding to (M, ω) when viewed as a nonsingular complete toric variety (the explicit relation appears in [9]). Because of this correspondence the search for their minimal model is reduced to an algebraic problem concerning fans associated to Delzant polytopes. If $2n \geqslant 6$ the problem is still too difficult but when $2n = 4$ the corresponding 2-dimensional fans have been classified; a proof of the following result, originally due to Oda in the 1970s, may also be found for instance in Fulton [10].

Theorem 1.2 (*Oda [11, Theorem 8.2]*)**.** *The inequivalent minimal models of symplectic toric manifolds are* $\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$, *and a Hirzebruch surface with parameter* $k \neq 1$.

The Delzant polytopes of the minimal models are: a simplex ($M = \mathbb{CP}^2$ with any multiple of the Fubini–Study form), a rectangle ($M = \mathbb{CP}^1 \times \mathbb{CP}^1$ with any product form), and a trapezoid (M a Hirzebruch surface, with one of its standard forms). The question is whether this classification can cover more cases.

Main Question. *What are the inequivalent minimal models of compact symplectic semitoric manifolds?*

Even more interesting would be to know whether the question can be answered as an application of the known invariants. However, it is not clear what the effect of blowing up and down is on the known invariants we have just mentioned. The image $F(M)$ is no longer necessarily a polygon, or even a convex set. The polygon P is obtained as the image of a homeomorphism $\varphi : F(M) \to P \subset \mathbb{R}^2$ which unfolds the singular affine structure of $F(M)$ into P, taking into account the monodromy (the construction of φ is delicate, see [7]). The effect of blowing up or down on P depends on the position of the focus–focus values of F, and here is where a new invariant of compact symplectic semitoric manifolds comes into play, we call it the *semitoric helix* and denote it by $\mathcal{H}$. Like in the toric case, $\mathcal{H}$ is given by (an equivalence class of) vectors in $\mathbb{Z}^2$, plus some additional information which we describe later more precisely and which includes the information of focus–focus singularities and monodromy (this does not appear in the toric case).

Analogous to the way in which from a Delzant polygon one constructs a fan, from P one constructs the helix $\mathcal{H}$ (after making some corrections related to the focus–focus singular points), see Fig. 4, though the helix can also be constructed directly from M, bypassing the polygon, as in Fig. 1. We describe the construction of $\mathcal{H}$ in detail in Section 4.1. The helix $\mathcal{H}$ contains the information encoding blowing up and blowing down, information which appears to be very difficult to extract from known invariants. And $\mathcal{H}$ generalizes the fan while taking into account the effects of the monodromy around the focus–focus singularities.[2] Moreover, $\mathcal{H}$ can be studied with algebraic techniques, and can be applied to prove the following, which is the main theorem of this paper.

[1] And in all important examples it is never a polygon, including the coupled spin–oscillator and the spin–orbit system.

[2] The helix is also related to the notion of *semitoric fan* introduced in [12], though they are not equivalent, the precise relation is discussed in Section 3.5.