Journal of Geometry and Physics 124 (2018) 16–54

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys

Painlevé equations, topological type property and reconstruction by the topological recursion

K. Iwaki [a,*], O. Marchal [b], A. Saenz [c]

[a] Graduate School of Mathematics, Nagoya University, Furocho, Nagoya, Japan
[b] Université de Lyon, CNRS UMR 5208, Université Jean Monnet, Institut Camille Jordan, France
[c] Department of Mathematics, University of Virginia, USA

ARTICLE INFO

Article history:
Received 9 October 2017
Accepted 18 October 2017
Available online 31 October 2017

MSC:
34M55
34M56
34E20
60B20

Keywords:
Painlevé equation
τ-function
Topological recursion
Determinantal formula
WKB method

ABSTRACT

In this article we prove that Lax pairs associated with $\hbar$-dependent six Painlevé equations satisfy the topological type property proposed by Bergère, Borot and Eynard for any generic choice of the monodromy parameters. Consequently we show that one can reconstruct the formal $\hbar$-expansion of the isomonodromic τ-function and of the determinantal formulas by applying the so-called topological recursion to the spectral curve attached to the Lax pair in all six Painlevé cases. Finally we illustrate the former results with the explicit computations of the first orders of the six τ-functions.

1. Introduction

It is now well known that the study of Hermitian random matrices is intrinsically related to integrable systems and Painlevé equations. In particular, it is proved that the partition functions of Hermitian matrix models coincide with the τ-functions of certain integrable systems [1,2]. Moreover, it is also well understood that the local correlations between eigenvalues in Hermitian random matrices exhibit universal behaviors when the size of the matrix goes to infinity. For example, the gap probability in the bulk of the distribution of eigenvalues can be directly connected to the Painlevé V equation, the so-called Hastings McLeod solution of the Painlevé II equation is related to the gap probability at the edge, and many similar results in different locations of the eigenvalues distribution are now available in relation with the other Painlevé equations. See [3–5] for example.

In a more algebraic perspective, it was realized that the connection between integrable systems and Hermitian matrix models can be understood because of the existence of loop equations (also known as Schwinger–Dyson equations) that can be solved perturbatively (under additional assumptions like convex potentials or genus 0 spectral curve) via the topological recursion introduced by Eynard and Orantin in [6]. Since the scope of the topological recursion has been proved to go much beyond matrix models, it is natural to wonder if one could define the equivalence of correlation functions arising in the formalism of the topological recursion directly into the integrable systems formalism. In [7] and [8], Bergère, Borot and

* Corresponding author.
E-mail addresses: iwaki@math.nagoya-u.ac.jp (K. Iwaki), olivier.marchal@univ-st-etienne.fr (O. Marchal), ais6a@virginia.edu (A. Saenz).

https://doi.org/10.1016/j.geomphys.2017.10.009
0393-0440/

Table 1.1
List of ($\hbar$-dependent) Painlevé equations. ($\dot{q} = \frac{dq}{dt}$.)

• (Painlevé I)

$$\hbar^2 \ddot{q} = 6q^2 + t. \tag{1.1}$$

• (Painlevé II)

$$\hbar^2 \ddot{q} = 2q^3 + tq - \theta + \frac{\hbar}{2}. \tag{1.2}$$

• (Painlevé III)

$$\hbar^2 \ddot{q} = \frac{\hbar^2}{q}\dot{q}^2 - \frac{\hbar^2}{t}\dot{q} + \frac{4}{t}\left(\theta_0 q^2 - \theta_\infty + \hbar\right) + 4q^3 - \frac{4}{q}. \tag{1.3}$$

• (Painlevé IV)

$$\hbar^2 \ddot{q} = \frac{\hbar^2}{2q}\dot{q}^2 + 2\left(3q^3 + 4tq^2 + \left(t^2 - 2\theta_\infty + \hbar\right)q - \frac{\theta_0^2}{q}\right). \tag{1.4}$$

• (Painlevé V)

$$\hbar^2 \ddot{q} = \tag{1.5}$$

$$\left(\frac{1}{2q} + \frac{1}{q-1}\right)(\hbar\dot{q})^2 - \hbar^2\frac{\dot{q}}{t} + \frac{(q-1)^2}{t^2}\left(\alpha q + \frac{\beta}{q}\right) + \frac{\gamma q}{t} + \frac{\delta q(q+1)}{q-1},$$

where

$$\alpha = \frac{(\theta_0 - \theta_1 - \theta_\infty)^2}{8}, \beta = -\frac{(\theta_0 - \theta_1 + \theta_\infty)^2}{8},$$

$$\gamma = \theta_0 + \theta_1 - \hbar \text{ and } \delta = -\frac{1}{2}.$$

• (Painlevé VI)

$$\begin{aligned}\hbar^2 \ddot{q} \;=\; & \frac{\hbar^2}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t}\right)\dot{q}^2 - \hbar^2\left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t}\right)\dot{q} \\ & + \frac{q(q-1)(q-t)}{t^2(t-1)^2}\left[\alpha + \beta\frac{t}{q^2} + \gamma\frac{t-1}{(q-1)^2} + \delta\frac{t(t-1)}{(q-t)^2}\right],\end{aligned} \tag{1.6}$$

where the parameters are

$$\alpha = \frac{1}{2}(\theta_\infty - \hbar)^2, \ \beta = -\frac{\theta_0^2}{2}, \ \gamma = \frac{\theta_1^2}{2} \text{ and } \delta = \frac{\hbar^2 - \theta_t^2}{2}.$$

Eynard suggested determinantal formulas that associate to any Lax pair (in fact any finite dimensional linear differential system) a set of correlation functions that satisfy the same loop equations as the one arising in Hermitian matrix models. Consequently at the perturbative level, one may expect these correlation functions to be reconstructed by the topological recursion. However, since loop equations may have many solutions it is not obvious why the functions generated by the topological recursion (that are one set of solutions to the loop equations) should necessarily identify with the determinantal formulas (that are also one set of solutions to the loop equations).

In [7] and [8] the authors discussed about sufficient conditions on the Lax pair with a small parameter $\hbar$, known as the topological type property, on the Lax pair to prove that both sets are identical. The purpose of this article is to show that the 2×2 Lax pairs associated with the six Painlevé equations with generic monodromy parameters satisfies the topological type property (see Definition 5.2).

We introduce the small parameter $\hbar$ in the Lax pairs presented by Jimbo, Miwa and Ueno in [9] through a rescaling of the parameters, providing a $\hbar$-dependent version of the Painlevé equations in Table 1.1 (whose standard forms can be recovered by setting $\hbar = 1$). Note that the Painlevé equations with a formal parameter has been discussed in terms of the WKB method in [10]. We also provide $\hbar$-dependent versions of the Hamiltonian structures underlying the Painlevé equations (cf. [11]). Then, we introduce and compute the spectral curve through the semi-classical limit of the Lax matrix. It turns out that the spectral curves arising in all six Lax matrices are of genus 0, and hence they admit a rational parametrization. Under some genericity assumptions, we verify that the spectral curves are generic so that the topological recursion algorithm can be applied. Then, our main results are formulated as follows (Theorem 5.4):

- Under some genericity assumption (Assumptions 3.2 and 4.2), we prove that the all Lax pairs associated with the six Painlevé equations are of topological type.
- Consequently, for all six Painlevé equations, Bergère–Borot–Eynard correlation functions defined from these Lax pairs coincide with the Eynard–Orantin differentials obtained from the application of the topological recursion to the corresponding spectral curve. This implies that the τ-function of the isomonodromy Lax system coincides with the generating function of the symplectic invariants of the spectral curve.

Our results extend similar results developed for the Painlevé II equation in [12] as well as partial results for the Painlevé V equation in [13]. Our analysis is sufficiently general to cover all six Painlevé equations.

We note that our approach is purely formal. The τ-functions reconstructed by the topological recursion correspond to formal power series solutions of the Painlevé equation. Such solutions are known to be divergent in general, and hence, our formal solutions are asymptotic expansion of exact solutions. Analytic properties of exact solution of Painlevé equations are widely studied; see [14] for example.