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Boson-fermion correspondence from factorization spaces

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A R T I C L E I N F O

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We give a proof of the boson-fermion correspondence, an isomorphism of lattice and fermion vertex algebras, in terms of a natural isomorphism of corresponding factorization spaces.

0. Introduction

0.1.

The notion of factorization space is a non-linear analog of the factorization algebra, which was introduced by Beilinson and Drinfeld [1] in their theory of chiral algebras. The notion of chiral algebra was introduced to give a geometric framework of vertex algebras, and to study the geometric Langlands correspondence [2].

A factorization space over a scheme X is a family of ind-schemes $\mathcal{G}_I$ over X^I for each finite set I with "factorization structure" given by isomorphisms between these $\mathcal{G}_I$'s. Given a factorization space on X, one can apply a linearization procedure, and if X is a smooth curve then one obtains a factorization algebra, an equivalent notion of chiral algebra.

Although the notion of factorization space looks highly complicated at first glance, it fits various kinds of moduli problems over algebraic curves very well. We may say that factorization spaces capture the intimate connection between two-dimensional conformal field theories and moduli problems on curves.

Along this line, one can ask a question: can one enhance various properties of vertex algebras to the level of factorization spaces? In this note, we consider this kind of problem for the boson-fermion correspondence.

Recall that the two-dimensional boson-fermion correspondence [3] is stated as an isomorphism

$$V_{\mathbb{Z}} \simeq \bigwedge \tag{0.1}$$

between the lattice vertex algebra $V_{\mathbb{Z}}$ attached to the lattice $\mathbb{Z}$ and the free fermion vertex super algebra $\bigwedge$. Here we borrowed the notations in [4]. The essence of the correspondence is that the vertex operators

$$V_{\pm}(z) :=: e^{\pm\varphi(z)} :, \quad \varphi(z) := q + a_0 \log(z) - \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{a_n}{n} z^{-n}$$

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acting on $V_{\mathbb{Z}}$ obey the same commutation relation with the fermionic operators $\psi(z)$, $\psi^*(z)$ acting on $\bigwedge$. Here a_n's and q denote the Heisenberg generators with the commutation relation $[a_m, a_n] = m\delta_{m+n,0}$ and $[q, a_n] = \delta_{n,0}$. Thus schematically we have

$$V_+(z) \longleftrightarrow \psi(z), \quad V_-(z) \longleftrightarrow \psi^*(z) \tag{0.2}$$

of vertex operators. See [4, §5.3] for the details.

Our main statement in this note is Theorem 5.1 giving an isomorphism

$$\mathcal{G}(X, \mathbb{Z}) \simeq \mathcal{G}r(X, \mathrm{SC}(1)) \tag{0.3}$$

between factorization spaces over a smooth curve X. Here $\mathcal{G}(X, \mathbb{Z})$ is the factorization space arising from the Picard variety of X (moduli space of line bundles on X). $\mathcal{G}r(X, \mathrm{SC}(1))$ is the Beilinson–Drinfeld Grassmannian for the special Clifford group $\mathrm{SC}(1)$. See Section 4.1 for the notations of Clifford groups.

These factorization spaces are equipped with factorization super linear bundles respectively, and the twisted linearization procedure gives the lattice and the Clifford chiral algebras. These chiral algebras correspond to the vertex algebras $V_{\mathbb{Z}}$ and $\bigwedge$ respectively. Thus the isomorphism (0.3) can be considered as an enhanced version of the boson-fermion correspondence (0.1).

0.2. Organization

Let us explain the organization of this article.

In Section 1 we recall the definitions of factorization spaces and factorization line bundles. We also recall the Beilinson–Drinfeld Grassmann $\mathcal{G}r(X, G)$ for a smooth algebraic curve X and a reductive group G. The space $\mathcal{G}r(X, G)$ is a standard example of factorization space,

Section 2 reviews the linearization procedure which produces a chiral algebra from factorization space.

In Section 3 we recall the factorization space $\mathcal{G}(X, \mathbb{Z})$ arises from the Picard functor. By the linearization procedure it gives the lattice vertex algebra $V_{\mathbb{Z}}$.

In Section 4 we consider the Beilinson–Drinfeld Grassmann $\mathcal{G}r(X, \mathrm{SC}(Q))$ for the Clifford group $\mathrm{SC}(Q)$ attached to the non-degenerate quadratic form Q. We explain that by the linearization procedure it yields the Clifford chiral algebra.

Section 5 gives the proof of the main theorem. The proof is not difficult, and the key observation is that we have a natural identification of moduli stacks

$$\mathcal{M}(X, \mathrm{SC}(1)) \simeq \mathrm{Pic}(X) \times \mathrm{Pic}(X)$$

of the moduli space of $\mathrm{SC}(1)$-bundles on X with two pieces of Picard varieties. It reflects the correspondence (0.2) of vertex operators $(V_+, V_-) \longleftrightarrow (\psi, \psi^*)$.

The ingredients in Sections 1–3 are basically known facts, and the main sources of our presentation are [1,4]. The content in Section 4 is also a variant of [4, Chap. 20].

0.3. Notation

We will work over the field $\mathbb{C}$ of complex numbers unless otherwise stated. The symbol $\otimes$ denotes the tensor product of $\mathbb{C}$-linear spaces.

For a scheme or an algebraic stack Z, we denote by $\mathcal{O}_Z$, Θ_Z, Ω_Z and $\mathcal{D}_Z$ the structure sheaf, the tangent sheaf, the sheaf of 1-forms and the sheaf of differential operators on Z respectively (if they are defined). By "an $\mathcal{O}$-module on Z" we mean a quasi-coherent sheaf on Z. By "a $\mathcal{D}$-module on Z" we mean a sheaf of *right* $\mathcal{D}_Z$-modules quasi-coherent as $\mathcal{O}_Z$-modules.

For a morphism $f : Z_1 \to Z_2$, the symbols $f^{\cdot}$ and $f_{\cdot}$ denote the inverse and direct image functors of $\mathcal{O}$-modules respectively.

Finally we will use the symbols $\mathcal{O} := k[[t]]$ for the algebra of formal series and $\mathcal{K} := k((t))$ for the field of formal Laurent series.

1. Factorization space

We follow [1, §3.10.16], [4, §20.4.1], [5, Chap. 5] and [6].

1.1. The category $\mathcal{S}$ of finite sets and surjections

The following category $\mathcal{S}$ will be used repeatedly.

Definition 1.1. *Let $\mathcal{S}$ be the category of non-empty finite sets and surjections. For $\pi : J \twoheadrightarrow I$ in $\mathcal{S}$ and $i \in I$ we set $J_i := \pi^{-1}(i) \subset J$.*