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Almost complex structures that are harmonic maps

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ABSTRACT

We find geometric conditions on a four-dimensional almost Hermitian manifold under which the almost complex structure is a harmonic map or a minimal isometric imbedding of the manifold into its twistor space.

1. Introduction

Recall that an almost complex structure on a Riemannian manifold (M, g) is called almost Hermitian if it is g-orthogonal. If a Riemannian manifold admits an almost Hermitian structure J, it has many such structures. One way to see this is to consider the twistor bundle $\pi : \mathcal{Z} \to M$ whose fibre at a point $p \in M$ consists of all g-orthogonal complex structures $I_p : T_pM \to T_pM$ ($I_p^2 = -Id$) on the tangent space of M at p yielding the same orientation as J_p. The fibre is the compact Hermitian symmetric space $SO(2n)/U(n)$ and its standard metric $-\frac{1}{2} Trace\, I_1 \circ I_2$ is Kähler–Einstein. The twistor space admits a natural Riemannian metric h such that the projection map $\pi : (\mathcal{Z}, h) \to (M, g)$ is a Riemannian submersion with totally geodesic fibres. Consider J as a section of the bundle $\pi : \mathcal{Z} \to M$ and take a section V with compact support K of the bundle $J^*\mathcal{V} \to M$, the pull-back under J of the vertical bundle $\mathcal{V} \to \mathcal{Z}$. There exists $\varepsilon > 0$ such that, for every point I of the compact set $J(K)$, the exponential map exp_I is a diffeomorphism of the ε-ball in $T_I\mathcal{Z}$. The function $\|V\|_h$ is bounded on M, so there exists a number $\varepsilon' > 0$ such that $\|tV(p)\|_h < \varepsilon$ for every $p \in M$ and $t \in (-\varepsilon', \varepsilon')$. Set $J_t(p) = exp_{J(p)}[tV(p)]$ for $p \in M$ and $t \in (-\varepsilon', \varepsilon')$. Then J_t is a section of $\mathcal{Z}$, i.e. an almost Hermitian structure on (M, g) (such that $J_t = J$ on $M \setminus K$).

Thus it is natural to seek for "reasonable" criteria that distinguish some of the almost Hermitian structures on a given Riemannian manifold (cf., for example, [1–4]). Motivated by the harmonic map theory, C. Wood [2,3] has suggested to consider as "optimal" those almost Hermitian structures $J : (M, g) \to (\mathcal{Z}, h)$ that are critical points of the energy functional under variations through sections of $\mathcal{Z}$, i.e. that are harmonic sections of the twistor bundle. In general, these critical points are not harmonic maps, but, by analogy, they are referred to as "harmonic almost complex structures" in [2,3]. It is more appropriate in the context of this article to call such structures "harmonic sections", a term used also in [2].

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The almost Hermitian structures that are critical points of the energy functional under variations through all maps $M \to \mathcal{Z}$ are genuine harmonic maps and the purpose of this paper is to find geometric conditions on a four-dimensional almost Hermitian manifold (M, g, J) under which the almost complex structure J is a harmonic map of (M, g) into $(\mathcal{Z}, h)$. We also find conditions for minimality of the submanifold $J(M)$ of the twistor space. As is well-known, in dimension four, there are three basic classes in the Gray–Hervella classification [5] of almost Hermitian structures—Hermitian, almost Kähler (symplectic) and Kähler structures. If (g, J) is Kähler, the map $J : (M, g) \to (\mathcal{Z}, h)$ is a totally geodesic isometric imbedding. In the case of a Hermitian structure, we express the conditions for harmonicity and minimality of J in terms of the Lee form, the Ricci and star-Ricci tensors of (M, g, J), while for an almost Kähler structure the conditions are in terms of the Ricci, star-Ricci and Nijenhuis tensors. Several examples illustrating these results are discussed in the last section of the paper, among them a Hermitian structure that is a harmonic section of the twistor bundle and a minimal isometric imbedding in it but not a harmonic map.

2. Preliminaries

Let (M, g) be an oriented Riemannian manifold of dimension four. The metric g induces a metric on the bundle of two-vectors $\pi : \Lambda^2TM \to M$ by the formula

$$g(v_1 \wedge v_2, v_3 \wedge v_4) = \frac{1}{2}det[g(v_i, v_j)].$$

The Levi-Civita connection of (M, g) determines a connection on the bundle Λ^2TM, both denoted by ∇, and the corresponding curvatures are related by

$$R(X \wedge Y)(Z \wedge T) = R(X, Y)Z \wedge T + Z \wedge R(X, Y)T$$

for $X, Y, Z, T \in TM$. The curvature operator $\mathcal{R}$ is the self-adjoint endomorphism of Λ^2TM defined by

$$g(\mathcal{R}(X \wedge Y), Z \wedge T) = g(R(X, Y)Z, T).$$

Let us note that we adopt the following definition for the curvature tensor R : $R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$.

The Hodge star operator defines an endomorphism $*$ of Λ^2TM with $*^2 = Id$. Hence we have the orthogonal decomposition

$$\Lambda^2TM = \Lambda^2_-TM \oplus \Lambda^2_+TM$$

where $\Lambda^2_\pm TM$ are the subbundles of Λ^2TM corresponding to the (±1)-eigenvalues of the operator $*$.

Let (E_1, E_2, E_3, E_4) be a local oriented orthonormal frame of TM. Set

$$s_1 = E_1 \wedge E_2 + E_3 \wedge E_4, \quad s_2 = E_1 \wedge E_3 + E_4 \wedge E_2, \quad s_3 = E_1 \wedge E_4 + E_2 \wedge E_3. \tag{1}$$

Then (s_1, s_2, s_3) is a local orthonormal frame of Λ^2_+TM defining an orientation on Λ^2_+TM, which does not depend on the choice of the frame (E_1, E_2, E_3, E_4).

For every $a \in \Lambda^2TM$, define a skew-symmetric endomorphism K_a of $T_{\pi(a)}M$ by

$$g(K_aX, Y) = 2g(a, X \wedge Y), \quad X, Y \in T_{\pi(a)}M. \tag{2}$$

Note that, denoting by G the standard metric $-\frac{1}{2}Trace\,PQ$ on the space of skew-symmetric endomorphisms, we have $G(K_a, K_b) = 2g(a, b)$ for $a, b \in \Lambda^2TM$. If $\sigma \in \Lambda^2_+TM$ is a unit vector, then K_σ is a complex structure on the vector space $T_{\pi(\sigma)}M$ compatible with the metric and the orientation of M. Conversely, the 2-vector σ dual to one half of the fundamental 2-form of such a complex structure is a unit vector in Λ^2_+TM. Thus the unit sphere subbundle $\mathcal{Z}$ of Λ^2_+TM parametrizes the complex structures on the tangent spaces of M compatible with its metric and orientation. This subbundle is called the twistor space of M.

The Levi-Civita connection ∇ of M preserves the bundles $\Lambda^2_\pm TM$, so it induces a metric connection on each of them denoted again by ∇. The horizontal distribution of Λ^2_+TM with respect to ∇ is tangent to the twistor space $\mathcal{Z}$. Thus we have the decomposition $T\mathcal{Z} = \mathcal{H} \oplus \mathcal{V}$ of the tangent bundle of $\mathcal{Z}$ into horizontal and vertical components. The vertical space $\mathcal{V}_\tau = \{V \in T_\tau\mathcal{Z} : \pi_*V = 0\}$ at a point $\tau \in \mathcal{Z}$ is the tangent space to the fibre of $\mathcal{Z}$ through τ. Considering $T_\tau\mathcal{Z}$ as a subspace of $T_\tau(\Lambda^2_+TM)$ (as we shall always do), $\mathcal{V}_\tau$ is the orthogonal complement of τ in $\Lambda^2_+T_{\pi(\tau)}M$. The map $V \ni \mathcal{V}_\tau \to K_V$ gives an identification of the vertical space with the space of skew-symmetric endomorphisms of $T_{\pi(\tau)}M$ that anti-commute with K_τ. Let s be a local section of $\mathcal{Z}$ such that $s(p) = \tau$ where $p = \pi(\tau)$. Considering s as a section of Λ^2_+TM, we have $\nabla_X s \in \mathcal{V}_\tau$ for every $X \in T_pM$ since s has a constant length. Moreover, $X^h_\tau = s_*X - \nabla_X s$ is the horizontal lift of X at τ.

Denote by $\times$ the usual vector cross product on the oriented 3-dimensional vector space $\Lambda^2_+T_pM$, $p \in M$, endowed with the metric g. Then it is easy to check that

$$g(R(a)b, c) = g(\mathcal{R}(b \times c), a) \tag{3}$$

for $a \in \Lambda^2T_pM$, $b, c \in \Lambda^2_+T_pM$. It is also easy to show that for every $a, b \in \Lambda^2_+T_pM$

$$K_a \circ K_b = -g(a, b)Id + K_{a\times b}. \tag{4}$$