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LAZY ORBITS: AN OPTIMIZATION PROBLEM ON THE SPHERE

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Abstract. Non-transitive subgroups of the orthogonal group play an important role in the non-Euclidean geometry. If G is a closed subgroup in the orthogonal group such that the orbit of a single Euclidean unit vector does not cover the (Euclidean) unit sphere centered at the origin then there always exists a non-Euclidean Minkowski functional such that the elements of G preserve the Minkowskian length of vectors. In other words the Minkowski geometry is an alternative of the Euclidean geometry for the subgroup G. It is rich of isometries if G is "close enough" to the orthogonal group or at least to one of its transitive subgroups. The measure of non-transitivity is related to the Hausdorff distances of the orbits under the elements of G to the Euclidean sphere. Its maximum/minimum belongs to the so-called lazy/busy orbits, i.e. they are the solutions of an optimization problem on the Euclidean sphere. The extremal distances allow us to characterize the reducible/irreducible subgroups. We also formulate an upper and a lower bound for the ratio of the extremal distances.

As another application of the analytic tools we introduce the rank of a closed non-transitive group G. We shall see that if G is of maximal rank then it is finite or reducible. Since the reducible and the finite subgroups form two natural prototypes of non-transitive subgroups, the rank seems to be a fundamental notion in their characterization. Closed, non-transitive groups of rank $n-1$ will be also characterized. Using the general results we classify all their possible types in lower dimensional cases $n=2,3$ and 4.

Finally we present some applications of the results to the holonomy group of a metric linear connection on a connected Riemannian manifold.

1. Introduction

In the preamble to his fourth problem presented at the International Mathematical Congress in Paris (1900) Hilbert suggested the examination of geometries standing next to Euclidean one in the sense that they satisfy much of Euclidean's axioms except some (tipically one) of them. In the classical non-Euclidean geometry the axiom taking to fail is the fameous parallel postulate. Another type of geometry standing next to Euclidean one is the geometry of normed spaces or, in a more general context, the geometry of Minkowski spaces [10]. The crucial test is not the parallelism but the congruence via the group of linear isometries. Consider the Euclidean coordinate space $\mathbb{R}^n$ and let $G \subset O(n)$ be a closed non-transitive subgroup in the orthogonal group. We are going to construct a compact convex body K containing the origin in its interior such that

(K1) K is not a unit ball with respect to any inner product (ellipsoid problem),
(K2) K is invariant under the elements of the subgroup G,
(K3) its boundary ∂K is a smooth hypersurface (regularity condition).

The Minkowski functional induced by K as a unit ball makes the vector space a non-Euclidean Minkowski space in the sense of condition (K1). The second condition (K2) says that G is a subgroup of the linear isometry group with respect to the Minkowski functional. In other words the Minkowski geometry is the alternative of the Euclidean geometry for the subgroup G. The regularity condition (K3) allows us to apply the standard differential geometric tools in this situation: the central role of the theory is played by the Hessian of the Minkowskian norm square [1]. In case of a differentiable manifold equipped with a Riemannian metric, the

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